(a) CrashHash is a cryptographic hash function invented by your colleague this morning. It zero-pads input $X$, splits it into $n$ 256-bit blocks $x_1 || x_2 || \ldots || x_n = X || 0^{(−|X|) \mod 256}$ and then appends a length-indicator block $x_{n+1} = \langle |X| \rangle$, as in the Merkle–Damgård construction. It then iterates a 512-bit to 256-bit compression function of the form $C(K, M) = E_K(M)$, where $E_K(M)$ is a blockcipher $E$ applied with 256-bit key $K$ to 256-bit message block $M$, as

$$z_1 = C(\langle 0 \rangle, x_1)$$
$$z_i = C(z_{i−1}, x_i) \quad (1 < i \leq n+1)$$

The value $H(X) = z_{n+1}$ is the hash value returned. Show that CrashHash is not collision resistant, even if $E$ is replaced with an ideal cipher. [6 marks]

(b) (i) How can one modify an implementation of the DES encryption function to obtain the decryption function? [4 marks]

(ii) Name two other features of DES that made it well suited for hardware implementation. [2 marks]

c) Your colleague has generated a set of $m = 200,000$ RSA key pairs that include a modulus $n_i = p_i q_i$ where $p_i$ and $q_i$ are 1536-bit prime numbers (for $1 \leq i \leq m$). The corresponding $p_i$ and $q_i$ values were discarded immediately after key generation and are no longer available.

Due to a bug in your colleague’s key-generation software, two types of fault have appeared in a random subset of the issued key pairs:

(i) For some key pairs $i$ we have $p_i = q_i$.

(ii) For some key pairs $i$ there exists another key pair $j$ in that set with $p_i = p_j$ and $i \neq j$.

Suggest practical tests that can identify all public keys affected by either of these problems and state how often the algorithms involved have to be executed for this task. [4 marks]

d) Calculate $7^{2000} \mod 100$ by hand. [4 marks]