6 Denotational Semantics (mpf23)

A right adjoint of a monotone function \( f : P \to Q \) between posets is a monotone function \( g : Q \to P \) such that \( \text{id}_P \subseteq g \circ f \) and \( f \circ g \subseteq \text{id}_Q \).

Let \( f : P \to Q \) be a monotone function with a right adjoint \( g : Q \to P \).

(a) For \( p \in P \) and \( q \in Q \), prove that \( f(p) \sqsubseteq q \) if, and only if, \( p \sqsubseteq g(q) \). [4 marks]

Let \( h : P \to P \) and \( \ell : Q \to Q \) be monotone functions such that \( f \circ h = \ell \circ f : P \to Q \).

(b) Prove that if \( h \) has a least pre-fixed point \( \text{fix}(h) \) then \( f(\text{fix}(h)) \) is a least pre-fixed point of \( \ell \). [8 marks]

Further assume that \( g \circ f = \text{id}_P \), in which case \( f \) is said to be an embedding and \( g \) a projection.

(c) Prove that if \( \ell \) has a least pre-fixed point \( \text{fix}(\ell) \) then \( g(\text{fix}(\ell)) \) is a least pre-fixed point of \( h \). [8 marks]