6 Denotational Semantics (mpf23)

A *right adjoint* of a monotone function $f : P \to Q$ between posets is a monotone function $g : Q \to P$ such that $\mathrm{id}_P \sqsubseteq g \circ f$ and $f \circ g \sqsubseteq \mathrm{id}_Q$.

Let $f : P \to Q$ be a monotone function with a right adjoint $g : Q \to P$.

(a) For $p \in P$ and $q \in Q$, prove that $f(p) \sqsubseteq_Q q$ if, and only if, $p \sqsubseteq_P g(q)$. [4 marks]

Let $h : P \to P$ and $\ell : Q \to Q$ be monotone functions such that $f \circ h = \ell \circ f : P \to Q$.

(b) Prove that if h has a least pre-fixed point $\mathit{fix}(h)$ then $f(\mathit{fix}(h))$ is a least pre-fixed point of ℓ. [8 marks]

Further assume that $g \circ f = \mathrm{id}_P$, in which case f is said to be an *embedding* and g a *projection*.

(c) Prove that if ℓ has a least pre-fixed point $\mathit{fix}(\ell)$ then $g(\mathit{fix}(\ell))$ is a least pre-fixed point of h. [8 marks]