6 Computation Theory (amp12)

A set $A$ equipped with a binary operation $@ : A \times A \rightarrow A$ is a **combinatory algebra** if there are elements $K, S \in A$ satisfying for all $a, b, c \in A$

\[
\begin{align*}
@(@(@K, a), b) &= a \\
@(@(@S, a), b, c) &= @(@(@a, c), @b, c))
\end{align*}
\]

(a) Show that there is a binary operation on the set of equivalence classes of closed $\lambda$-terms for the equivalence relation of $\beta$-conversion that makes it a combinatory algebra. [5 marks]

(b) Show that every combinatory algebra $A$ contains an element $I$ satisfying

\[ @I, a \] = a \]

for all $a \in A$. [Hint: what does (2) tell us when $a = b = K$?] [2 marks]

(c) For an arbitrary combinatory algebra $A$, let $A[x]$ denote the set of expressions given by the grammar

\[ e ::= x \mid \lbrack a \rbrack \mid (ee) \]

where $x$ is some fixed symbol not in $A$ and $a$ ranges over the elements of $A$. Given $e \in A[x]$ and $a \in A$, let $e[x := a]$ denote the element of $A$ resulting from interpreting occurrences of $x$ in $e$ by $a$, interpreting the expressions of the form $\lbrack a \rbrack$ by $a'$ and interpreting expressions of the form $(ee')$ using $@$.

\[ (i) \] Give the clauses in a definition of $e[x := a]$ by recursion on the structure of $e$. [2 marks]

\[ (ii) \] For each $e \in A[x]$ show how to define an element $\Lambda_x e \in A$ with the property that

\[ @(@\Lambda_x e, a) = e[x := a] \]

for all $a \in A$. [6 marks]

(d) Recall the usual encoding of Booleans in $\lambda$-calculus. Using Part (c)(ii), show that in any combinatory algebra $A$ there are elements $\text{True}, \text{False} \in A$ and a function $\text{If} : A \times A \rightarrow A$ satisfying

\[
\begin{align*}
@(@\text{If}(a, b), \text{True}) &= a \\
@(@\text{If}(a, b), \text{False}) &= b
\end{align*}
\]

for all $a, b \in A$ [5 marks]