5 Computation Theory (amp12)

(*a*) For each $n, e \in \mathbb{N}$, let $\varphi_e^{(n)}$ denote the partial function $\mathbb{N}^n \rightharpoonup \mathbb{N}$ computed by the register machine with index e using registers $\mathtt{R}_1, \ldots, \mathtt{R}_n$ to store the n arguments and register $\mathtt{R}_0$ to store the result, if any.

Explain why for each $m, n \in \mathbb{N}$ there is a totally defined register machine computable function $S_{m,n} : \mathbb{N}^{1+m} \to \mathbb{N}$ with the property that for all $(e, \vec{x}) \in \mathbb{N}^{1+m}$ and $\vec{y} \in \mathbb{N}^n$

$$\varphi^{(n)}_{S_{m,n}(e,\vec{x})}(\vec{y}) \equiv \varphi_e^{(m+n)}(\vec{x}, \vec{y}) \tag{1}$$

where $\equiv$ denotes Kleene equivalence: for all $z \in \mathbb{N}$, the left-hand side is defined and equal to z if and only if the right-hand side is defined and equal to z. Your explanation should make clear what assumptions you are making about the relationship between numbers and register machine programs. [10 marks]

(*b*) Let $f : \mathbb{N}^{1+m+n} \rightharpoonup \mathbb{N}$ be a register machine computable partial function of $1+m+n$ arguments for some $m, n \in \mathbb{N}$.

(*i*) Why is the partial function $\hat{f} : \mathbb{N}^{1+m+n} \rightharpoonup \mathbb{N}$ satisfying for all $(z, \vec{x}, \vec{y}) \in \mathbb{N}^{1+m+n}$

$$\hat{f}(z, \vec{x}, \vec{y}) \equiv f(S_{1+m,n}(z, z, \vec{x}), \vec{x}, \vec{y}) \tag{2}$$

register machine computable? [3 marks]

(*ii*) By considering $S_{1+m,n}(e, e, \vec{x})$ where e is an index for the partial function $\hat{f}$ in part (*b*)(*i*), prove that there is a totally-defined register machine computable function $\mathsf{fix}\, f : \mathbb{N}^m \to \mathbb{N}$ with the property that for all $\vec{x} \in \mathbb{N}^m$ and $\vec{y} \in \mathbb{N}^n$

$$\varphi^{(n)}_{\mathsf{fix}\, f(\vec{x})}(\vec{y}) \equiv f(\mathsf{fix}\, f(\vec{x}), \vec{x}, \vec{y}) \tag{3}$$

[7 marks]