5 Computation Theory (amp12)

(a) For each \( n, e \in \mathbb{N} \), let \( \varphi_e^{(n)} \) denote the partial function \( \mathbb{N}^n \to \mathbb{N} \) computed by the register machine with index \( e \) using registers \( R_1, \ldots, R_n \) to store the \( n \) arguments and register \( R_0 \) to store the result, if any.

Explain why for each \( m, n \in \mathbb{N} \) there is a totally defined register machine computable function \( S_{m,n} : \mathbb{N}^{1+m} \to \mathbb{N} \) with the property that for all \( (e, \vec{x}) \in \mathbb{N}^{1+m} \) and \( \vec{y} \in \mathbb{N}^n \)

\[
\varphi_{S_{m,n}(e,\vec{x})}^{(n)}(\vec{y}) \equiv \varphi_e^{(m+n)}(\vec{x}, \vec{y})
\]

where \( \equiv \) denotes Kleene equivalence: for all \( z \in \mathbb{N} \), the left-hand side is defined and equal to \( z \) if and only if the right-hand side is defined and equal to \( z \). Your explanation should make clear what assumptions you are making about the relationship between numbers and register machine programs. [10 marks]

(b) Let \( f : \mathbb{N}^{1+m+n} \to \mathbb{N} \) be a register machine computable partial function of \( 1+m+n \) arguments for some \( m, n \in \mathbb{N} \).

(i) Why is the partial function \( \hat{f} : \mathbb{N}^{1+m+n} \to \mathbb{N} \) satisfying for all \( (z, \vec{x}, \vec{y}) \in \mathbb{N}^{1+m+n} \)

\[
\hat{f}(z, \vec{x}, \vec{y}) \equiv f(S_{1+m,n}(z, z, \vec{x}), \vec{x}, \vec{y})
\]

register machine computable? [3 marks]

(ii) By considering \( S_{1+m,n}(e, e, \vec{x}) \) where \( e \) is an index for the partial function \( \hat{f} \) in part (b)(i), prove that there is a totally-defined register machine computable function \( \text{fix} f : \mathbb{N}^m \to \mathbb{N} \) with the property that for all \( \vec{x} \in \mathbb{N}^m \) and \( \vec{y} \in \mathbb{N}^n \)

\[
\varphi_{\text{fix} f(\vec{x})}^{(n)}(\vec{y}) \equiv f(\text{fix} f(\vec{x}), \vec{x}, \vec{y})
\]

[7 marks]