4 Complexity Theory (mpf23)

(a) For a complexity class $C$, let $\text{co-} C = \{ \overline{L} \mid L \in C \}$ and say that $C$ is closed under complementation whenever $C = \text{co-} C$.

Argue as to whether the following statements are true, false, or unknown.

(i) All deterministic time complexity classes are closed under complementation. [3 marks]

(ii) All non-deterministic time complexity classes are closed under complementation. [3 marks]

(b) For a mapping $f : \Sigma \rightarrow \Sigma$ on an alphabet $\Sigma$ and a language $L \subseteq \Sigma^*$, define $f[L] = \{ f^\sharp(w) \in \Sigma^* \mid w \in L \}$ where $f^\sharp(a_1 \cdots a_n) = f(a_1) \cdots f(a_n)$.

Prove that $L \in \text{NP}$ implies $f[L] \in \text{NP}$. [4 marks]

(c) Consider the following decision problem.

Q: Given natural numbers $m$ and $n$ in $\mathbb{N}$, and bits $a_{i,j}^{(k)}$ and $b_k$ in $\{0, 1\}$ for $1 \leq k \leq m$ and $1 \leq i, j \leq n$, determine whether the system of equations $\sum_{1 \leq i, j \leq n} a_{i,j}^{(k)} x_i x_j = b_k \ (1 \leq k \leq m)$ with unknowns $x_1, \ldots, x_n$ has a solution in arithmetic modulo 2.

(i) Prove that Q is in NP. [3 marks]

(ii) By means of a polynomial-time reduction from the problem 3CNF, or otherwise, prove that Q is NP-hard. [Hint: Note, for instance, that $x = \overline{y}$ in the Boolean algebra $\{0, 1\}$ if, and only if, $xx + yy = 1$ in arithmetic modulo 2.] [7 marks]

You may use standard results provided that you state them clearly.