4 Complexity Theory (mpf23)

(*a*) For a complexity class $\mathcal{C}$, let co-$\mathcal{C} = \{\, \overline{L} \mid L \in \mathcal{C} \,\}$ and say that $\mathcal{C}$ is closed under complementation whenever $\mathcal{C} =$ co-$\mathcal{C}$.

Argue as to whether the following statements are true, false, or unknown.

(*i*) All deterministic time complexity classes are closed under complementation. [3 marks]

(*ii*) All non-deterministic time complexity classes are closed under complementation. [3 marks]

(*b*) For a mapping $f : \Sigma \to \Sigma$ on an alphabet Σ and a language $L \subseteq \Sigma^*$, define $f[L] = \{\, f^\natural(w) \in \Sigma^* \mid w \in L \,\}$ where $f^\natural(a_1 \cdots a_n) = f(a_1) \cdots f(a_n)$.

Prove that $L \in \mathrm{NP}$ implies $f[L] \in \mathrm{NP}$. [4 marks]

(*c*) Consider the following decision problem.

> Q: Given natural numbers m and n in $\mathbb{N}$, and bits $a^{(k)}_{i,j}$ and b_k in $\{0,1\}$ for $1 \leq k \leq m$ and $1 \leq i,j \leq n$, determine whether the system of equations $\sum_{1\leq i,j\leq n} a^{(k)}_{i,j}\, x_i\, x_j = b_k$ $(1 \leq k \leq m)$ with unknowns $x_1, \ldots, x_n$ has a solution in arithmetic modulo 2.

(*i*) Prove that Q is in NP. [3 marks]

(*ii*) By means of a polynomial-time reduction from the problem 3CNF, or otherwise, prove that Q is NP-hard. [*Hint:* Note, for instance, that $x = \neg y$ in the Boolean algebra $\{0,1\}$ if, and only if, $xx + yy = 1$ in arithmetic modulo 2.] [7 marks]

You may use standard results provided that you state them clearly.