8 Discrete Mathematics (mpf23)

(*a*) Let i and j be positive integers.

(*i*) Prove that there exist natural numbers a and b such that $a \cdot i = b \cdot j + \mathrm{gcd}(i, j)$. You may use standard results provided that you state them clearly.
[4 marks]

(*ii*) Let m be a positive integer. Prove that, for all integers n,

$$(\ n^i \equiv 1 \pmod{m} \ \wedge \ n^j \equiv 1 \pmod{m}\ ) \implies n^{\mathrm{gcd}(i,j)} \equiv 1 \pmod{m}$$

[3 marks]

(*b*) (*i*) For sets A and B, let $\approx$ be the binary relation on $(A \Rightarrow B)$ defined, for all $f, g \in (A \Rightarrow B)$, by

$$f \approx g \iff \exists\, \alpha \in \mathrm{Bij}(A, A).\, \exists\, \beta \in \mathrm{Bij}(B, B).\ \beta \circ f = g \circ \alpha$$

Prove that $\approx$ is an equivalence relation on $(A \Rightarrow B)$. [9 marks]

(*ii*) Recalling that, for $n \in \mathbb{N}$, we let $[n] = \{\, i \in \mathbb{N} \mid 0 \leq i < n \,\}$, define

$$S_n \ = \ \big([n] \Rightarrow [2] \big)_{/\approx}$$

that is, the set S_n is the quotient of $\big([n] \Rightarrow [2] \big)$ under the equivalence relation $\approx$.

(A) List the elements of S_n for each $n \in [4]$. [2 marks]

(B) What is the cardinality of S_n for each $n \in \mathbb{N}$? [2 marks]