(a) Let $i$ and $j$ be positive integers.

(i) Prove that there exist natural numbers $a$ and $b$ such that $a \cdot i = b \cdot j + \gcd(i, j)$. You may use standard results provided that you state them clearly. [4 marks]

(ii) Let $m$ be a positive integer. Prove that, for all integers $n$,

$$n^i \equiv 1 \pmod{m} \land n^j \equiv 1 \pmod{m} \implies n^{\gcd(i,j)} \equiv 1 \pmod{m}$$

[3 marks]

(b) (i) For sets $A$ and $B$, let $\approx$ be the binary relation on $(A \Rightarrow B)$ defined, for all $f, g \in (A \Rightarrow B)$, by

$$f \approx g \iff \exists \alpha \in \text{Bij}(A, A). \exists \beta \in \text{Bij}(B, B). \beta \circ f = g \circ \alpha$$

Prove that $\approx$ is an equivalence relation on $(A \Rightarrow B)$. [9 marks]

(ii) Recalling that, for $n \in \mathbb{N}$, we let $[n] = \{ i \in \mathbb{N} \mid 0 \leq i < n \}$, define

$$S_n = ([n] \Rightarrow [2]) / \approx$$

that is, the set $S_n$ is the quotient of $( [n] \Rightarrow [2] )$ under the equivalence relation $\approx$.

(A) List the elements of $S_n$ for each $n \in [4]$. [2 marks]

(B) What is the cardinality of $S_n$ for each $n \in \mathbb{N}$? [2 marks]