9 Algorithms (djw1005)

Consider a directed graph in which each edge is labelled by a pair of non-negative costs, for example a distance and a travel time.

We say that a path with costs \((c_a, c_b)\) is Pareto dominated if there is another path with the same start and end vertices and with costs \((c'_a, c'_b)\) such that \(c'_a \leq c_a\) and \(c'_b \leq c_b\) and at least one of these inequalities is strict. A path is called Pareto efficient if it is not Pareto dominated by any other path. (These concepts are named after the economist Vilfredo Pareto.)

(a) In the graph shown here, find all Pareto efficient paths from \(s\) to \(t\), and state their costs. [1 mark]

(b) Show that, if \(v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k\) is a Pareto efficient path from \(v_0\) to \(v_k\), then \(v_0 \rightarrow \cdots \rightarrow v_{k-1}\) is a Pareto efficient path from \(v_0\) to \(v_{k-1}\). [3 marks]

(c) Let \(v_0 \rightarrow \cdots \rightarrow v_k\) be a Pareto efficient path from \(v_0\) to \(v_k\), and let its costs be \((c_a, c_b)\). Show that there is a Pareto efficient path from \(v_0\) to \(v_k\) with costs \((c_a, c_b)\) that has \(\leq V - 1\) edges, where \(V\) is the number of vertices in the graph. [3 marks]

(d) We are given a start vertex \(s\). Give an algorithm to compute all costs achievable by Pareto efficient paths from \(s\) to every other vertex. [6 marks]

(e) Prove that your algorithm is correct. [7 marks]

[Note: The version of this question that appeared in the exam contained an error, which has now been corrected.]