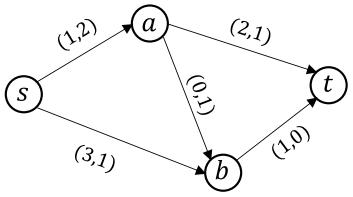
COMPUTER SCIENCE TRIPOS Part IA – 2021 – Paper 1

9 Algorithms (djw1005)

Consider a directed graph in which each edge is labelled by a pair of non-negative costs, for example a distance and a travel time.

We say that a path with costs (c_a, c_b) is *Pareto dominated* if there is another path with the same start and end vertices and with costs (c'_a, c'_b) such that $c'_a \leq c_a$ and $c'_b \leq c_b$ and at least one of these inequalities is strict. A path is called *Pareto efficient* if it is not Pareto dominated by any other path. (These concepts are named after the economist Vilfredo Pareto.)



(*a*) In the graph shown here, find all Pareto efficient paths from s to t, and state their costs. [1 mark]

(*b*) Show that, if $v_0 \to v_1 \to \cdots \to v_k$ is a Pareto efficient path from v_0 to v_k, then $v_0 \to \cdots \to v_{k-1}$ is a Pareto efficient path from v_0 to v_{k-1}. [3 marks]

(*c*) Let $v_0 \to \cdots \to v_k$ be a Pareto efficient path from v_0 to v_k, and let its costs be (c_a, c_b). Show that there is a Pareto efficient path from v_0 to v_k with costs (c_a, c_b) that has $\leq V - 1$ edges, where V is the number of vertices in the graph. [3 marks]

(*d*) We are given a start vertex s. Give an algorithm to compute *all* costs achievable by Pareto efficient paths from s to every other vertex. [6 marks]

(*e*) Prove that your algorithm is correct. [7 marks]

[*Note:* The version of this question that appeared in the exam contained an error, which has now been corrected.]