**6 Denotational Semantics (mpf23)**

($a$) For a poset $(P, \sqsubseteq)$, the *join* of $x, y \in P$ is defined to be the element $x \sqcup y \in P$ such that

$$\forall\, p \in P.\; x \sqcup y \sqsubseteq p \iff (\, x \sqsubseteq p \wedge y \sqsubseteq p\,)$$

A poset is said to be *join complete* if every pair of elements in it has a join.

For a join-complete cpo $D$, show that the function $\sqcup : D \times D \to D$ mapping $(x, y) \in D \times D$ to $x \sqcup y \in D$ is continuous. [8 marks]

($b$) Let $(D, \sqsubseteq)$ be a domain.

($i$) For a continuous function $f : D \to D$, prove that the subset of $D$

$$\widehat{f} \;=\; \{\, x \in D \mid f(x) \sqsubseteq x \,\}$$

ordered by $\sqsubseteq$ is a domain. [6 marks]

($ii$) For $d \in D$, let $\uparrow\!(d) = \{\, x \in D \mid d \sqsubseteq x \,\}$.

For a continuous function $g : D \to D$, prove that if $(D, \sqsubseteq)$ is join complete then, for all $d \in D$, the subset of $D$

$$\uparrow\!(d) \,\cap\, \widehat{g}$$

ordered by $\sqsubseteq$ is a domain. [6 marks]