9 Discrete Mathematics (mpf23)

(a) A partition of a set U is a family of sets $\mathcal{H} \subseteq \mathcal{P}(U)$ such that

- $\forall\, B \in \mathcal{H}.\ B \neq \emptyset$
- $\forall\, A, B \in \mathcal{H}.\ A \cap B \neq \emptyset \implies A = B$
- $U \subseteq \bigcup \mathcal{H}$

Prove that if $\mathcal{F}$ is a partition of a set A and $\mathcal{G}$ is a partition of a set B then

$$\mathcal{F} \otimes \mathcal{G} = \{\, Z \in \mathcal{P}(A \times B) \mid \exists\, X \in \mathcal{F}.\ \exists\, Y \in \mathcal{G}.\ Z = X \times Y \,\}$$

is a partition of the set $A \times B$. [6 marks]

(b) Let U be a set and $f : \mathcal{P}(U) \to \mathcal{P}(U)$ be a function such that

$$\forall\, X, Y \in \mathcal{P}(U)\,.\ X \subseteq Y \implies f(X) \subseteq f(Y)$$

Define $\mathcal{F} = \{\, Z \in \mathcal{P}(U) \mid f(Z) \subseteq Z \,\}$ and $\Phi = \bigcap \mathcal{F}$.

(i) Prove that $\Phi \in \mathcal{F}$. [4 marks]

(ii) Prove that $f(\Phi) \in \mathcal{F}$. [2 marks]

(iii) Prove that $f(\Phi) = \Phi$. [2 marks]

(c) Define without proof a bijection from $\mathbb{N}$ to $\{\, \mathtt{0}, \mathtt{1} \,\}^*$ and its inverse. [6 marks]