6 Computation Theory (amp12)

(a) (i) Give an inductive definition of the relation $M =_\beta N$ of $\beta$-conversion between $\lambda$-terms $M$ and $N$. [3 marks]

(ii) What is meant by a term in $\beta$-normal form? [1 mark]

(iii) If $M$ and $N$ are in $\beta$-normal form, explain why $M =_\beta N$ implies that $M$ and $N$ are $\alpha$-equivalent $\lambda$-terms. [2 marks]

(You need not define notions such as $\alpha$-equivalence and capture-avoiding substitution.)

(b) Show that there are $\lambda$-terms $\text{True}$, $\text{False}$ and $\text{If}$ satisfying $\text{If } \text{True} \ N =_\beta M$ and $\text{If } \text{False} \ N =_\beta N$ for all $\lambda$-terms $M$ and $N$ and with $\text{True} \neq _\beta \text{False}$. [4 marks]

(c) Define Curry’s fixed point combinator $Y$ and prove its fixed point property. [3 marks]

(d) Consider the following two properties of a $\lambda$-term $M$:

(I) there exist $\lambda$-terms $A$ and $B$ with $M A =_\beta \text{True}$ and $M B =_\beta \text{False}$

(II) for all $\lambda$-terms $N$, either $M N =_\beta \text{True}$ or $M N =_\beta \text{False}$.

Prove that $M$ cannot have both properties (I) and (II). [Hint: if $M$ has property (I), consider $M (Y(\lambda x. \text{If } (M x) B A))$.] [4 marks]

(e) Deduce that there is no $\lambda$-term $E$ such that for all $\lambda$-terms $M$ and $N$

$$E M N =_\beta \begin{cases} \text{True} & \text{if } M =_\beta N \\ \text{False} & \text{otherwise} \end{cases}$$

[3 marks]