**6 Computation Theory (amp12)**

(*a*) (*i*) Give an inductive definition of the relation $M =_\beta N$ of *β-conversion* between λ-terms $M$ and $N$. [3 marks]

(*ii*) What is meant by a term in *β-normal form*? [1 mark]

(*iii*) If $M$ and $N$ are in β-normal form, explain why $M =_\beta N$ implies that $M$ and $N$ are α-equivalent λ-terms. [2 marks]

(You need not define notons such as α-equivalence and capture-avoiding substitution.)

(*b*) Show that there are λ-terms $\mathsf{True}$, $\mathsf{False}$ and $\mathsf{If}$ satisfying $\mathsf{If}\,\mathsf{True}\,M\,N =_\beta M$ and $\mathsf{If}\,\mathsf{False}\,M\,N =_\beta N$ for all λ-terms $M$ and $N$ and with $\mathsf{True} \neq_\beta \mathsf{False}$. [4 marks]

(*c*) Define *Curry's fixed point combinator* $\mathsf{Y}$ and prove its fixed point property. [3 marks]

(*d*) Consider the following two properties of a λ-term $M$:

(I) there exist λ-terms $A$ and $B$ with $M\,A =_\beta \mathsf{True}$ and $M\,B =_\beta \mathsf{False}$

(II) for all λ-terms $N$, either $M\,N =_\beta \mathsf{True}$ or $M\,N =_\beta \mathsf{False}$.

Prove that $M$ cannot have both properties (I) and (II). [Hint: if $M$ has property (I), consider $M\,(Y(\lambda x.\,\mathsf{If}\,(M\,x)\,B\,A))$.] [4 marks]

(*e*) Deduce that there is no λ-term $E$ such that for all λ-terms $M$ and $N$

$$E\,M\,N =_\beta \begin{cases} \mathsf{True} & \text{if } M =_\beta N \\ \mathsf{False} & \text{otherwise} \end{cases}$$

[3 marks]