6 Computation Theory (amp12)

(*a*) (*i*) Give an inductive definition of the relation $M =_\beta N$ of *β-conversion* between λ-terms M and N. [3 marks]

(*ii*) What is meant by a term in *β-normal form*? [1 mark]

(*iii*) If M and N are in β-normal form, explain why $M =_\beta N$ implies that M and N are α-equivalent λ-terms. [2 marks]

(You need not define notons such as α-equivalence and capture-avoiding substitution.)

(*b*) Show that there are λ-terms True, False and If satisfying $\mathsf{If}\,\mathsf{True}\,M\,N =_\beta M$ and $\mathsf{If}\,\mathsf{False}\,M\,N =_\beta N$ for all λ-terms M and N and with $\mathsf{True} \neq_\beta \mathsf{False}$. [4 marks]

(*c*) Define *Curry's fixed point combinator* Y and prove its fixed point property. [3 marks]

(*d*) Consider the following two properties of a λ-term M:

(I) there exist λ-terms A and B with $M\,A =_\beta \mathsf{True}$ and $M\,B =_\beta \mathsf{False}$

(II) for all λ-terms N, either $M\,N =_\beta \mathsf{True}$ or $M\,N =_\beta \mathsf{False}$.

Prove that M cannot have both properties (I) and (II). [Hint: if M has property (I), consider $M\,(Y(\lambda x.\,\mathsf{If}\,(M\,x)\,B\,A))$.] [4 marks]

(*e*) Deduce that there is no λ-term E such that for all λ-terms M and N

$$E\,M\,N =_\beta \begin{cases} \mathsf{True} & \text{if } M =_\beta N \\ \mathsf{False} & \text{otherwise} \end{cases}$$

[3 marks]