4 Complexity Theory (ad260)

A Boolean formula $\phi$ is in conjunctive normal form (CNF) if it is the conjunction of clauses, each of which is the disjunction of literals. It is said to be in $k$-CNF (for $k \in \mathbb{N}$) if each clause has exactly $k$ literals in it.

An assignment $\sigma : V \to \{\text{true}, \text{false}\}$ of truth values to the variables is a satisfying assignment for a CNF formula $\phi$ if it makes at least one literal in each clause of $\phi$ true. It is said to be a not-all-equals assignment for $\phi$ if it makes at least one literal in each clause of $\phi$ true and at least one literal in each clause of $\phi$ false.

Let CNF-SAT denote the problem of determining, given a formula in CNF, whether it has a satisfying assignment.

Let $k$-SAT denote the problem of determining, given a formula in $k$-CNF, whether it has a satisfying assignment.

Let $k$-NAE denote the problem of determining, given a formula in $k$-CNF, whether it has a not-all-equals assignment.

(a) Explain why CNF-SAT is NP-complete. Your explanation should include a full definition of NP-completeness and a brief sketch of the proof of the Cook-Levin theorem. [5 marks]

(b) Show that 3-SAT is NP-complete by means of a suitable reduction. [3 marks]

(c) Give a polynomial-time reduction from 3-SAT to 4-NAE. What can you conclude about the complexity of the latter problem? (Hint: consider introducing one new variable and adding it to every clause.) [8 marks]

(d) Show that the problem 3-NAE is NP-complete. (Hint: consider a reduction from 4-NAE) [4 marks]