3 Complexity Theory (ad260)

A Boolean formula $\phi$ is said to be satisfiable if there is an assignment $\sigma : V \rightarrow \{\text{true}, \text{false}\}$ of values to the variables of $\phi$ that makes it true.

A quantified Boolean formula $\theta$ is an expression that is (i) either a Boolean formula; or (ii) $\exists X \phi$ where $\phi$ is a quantified Boolean formula and $X$ is variable; or (iii) $\forall X \phi$ where $\phi$ is a quantified Boolean formula and $X$ is variable.

We say that a quantified Boolean formula $\theta$ is satisfied by an assignment $\sigma : V \rightarrow \{\text{true}, \text{false}\}$ if either

- $\theta$ is a Boolean formula that is made true by $\sigma$; or
- $\theta$ is $\exists X \phi$ and either $\sigma[X/\text{true}]$ or $\sigma[X/\text{false}]$ make $\phi$ true; or
- $\theta$ is $\forall X \phi$ and both $\sigma[X/\text{true}]$ and $\sigma[X/\text{false}]$ make $\phi$ true.

Here, $\sigma[X/v]$ denotes the assignment that is the same as $\sigma$ for all variables apart from $X$, and it maps $X$ to the truth value $v$.

We write QBF for the decision problem of determining whether a given quantified Boolean formula is satisfiable. In answering the questions below, you may assume the NP-completeness of any standard problem, as long as you state your assumptions clearly.

(a) Show that QBF is NP-hard. [4 marks]

(b) Show that QBF is co-NP-hard. [6 marks]

(c) Show that QBF is in \textsc{PSPACE}. [6 marks]

(d) Is QBF NP-complete? Why or why not? [4 marks]