3 Complexity Theory (ad260)

A Boolean formula ϕ is said to be *satisfiable* if there is an assignment $\sigma : V \to \{\text{true}, \text{false}\}$ of values to the variables of ϕ that makes it true.

A *quantified Boolean formula* θ is an expression that is *(i)* either a Boolean formula; or *(ii)* $\exists X \phi$ where ϕ is a quantified Boolean formula and X is variable; or *(iii)* $\forall X \phi$ where ϕ is a quantified Boolean formula and X is variable.

We say that a quantified Boolean formula θ is satisfied by an assignment $\sigma : V \to \{\text{true}, \text{false}\}$ if either

- θ is a Boolean formula that is made true by σ; or
- θ is $\exists X \phi$ and either $\sigma[X/\text{true}]$ or $\sigma[X/\text{false}]$ make ϕ true; or
- θ is $\forall X \phi$ and both $\sigma[X/\text{true}]$ and $\sigma[X/\text{false}]$ make ϕ true.

Here, $\sigma[X/v]$ denotes the assignment that is the same as σ for all variables apart from X, and it maps X to the truth value v.

We write QBF for the decision problem of determining whether a given quantified Boolean formula is satisfiable. In answering the questions below, you may assume the NP-completeness of any standard problem, as long as you state your assumptions clearly.

(a) Show that QBF is NP-hard. [4 marks]

(b) Show that QBF is co-NP-hard. [6 marks]

(c) Show that QBF is in PSPACE. [6 marks]

(d) Is QBF NP-complete? Why or why not? [4 marks]