8 Discrete Mathematics (gw104)

A binary relation $\prec$ on a set A is *well-founded* iff there are no infinite descending chains $\cdots \prec a_i \prec \cdots \prec a_1 \prec a_0$.

(a) Show a binary relation $\prec$ on a set A is well-founded iff any nonempty subset Q of A has a minimal element, *i.e.* an element m such that

$$m \in Q \ \wedge \ \forall b \prec m.\, b \notin Q \ .$$

[5 marks]

(b) Show that defining

$$(n_1, n_2) \prec (n'_1, n'_2) \ \Leftrightarrow \ (n_1, n_2) \neq (n'_1, n'_2) \text{ and } n_1 \leq n'_1 \text{ and } n_2 \leq n'_2$$

determines a well-founded relation between pairs of positive natural numbers.

[7 marks]

(c) Let $\longrightarrow$ be a binary relation between pairs of positive natural numbers for which

$$(m, n) \longrightarrow (m, n - m) \text{ if } m < n\,, \quad \text{and} \quad (m, n) \longrightarrow (m - n, n) \text{ if } n < m\ .$$

Using (a) and (b), or otherwise, show that for all pairs of positive natural numbers (m, n), there is a natural number h such that

$$(m, n) \longrightarrow^* (h, h)\ .$$

[8 marks]