8 Discrete Mathematics (gw104)

A binary relation \( \prec \) on a set \( A \) is *well-founded* iff there are no infinite descending chains \( \cdots \prec a_i \prec \cdots \prec a_1 \prec a_0 \).

(a) Show a binary relation \( \prec \) on a set \( A \) is well-founded iff any nonempty subset \( Q \) of \( A \) has a minimal element, *i.e.* an element \( m \) such that

\[
m \in Q \land \forall b \prec m. \ b \notin Q .
\]

[5 marks]

(b) Show that defining

\[
(n_1, n_2) \prec (n'_1, n'_2) \iff (n_1, n_2) \neq (n'_1, n'_2) \text{ and } n_1 \leq n'_1 \text{ and } n_2 \leq n'_2
\]

determines a well-founded relation between pairs of positive natural numbers.

[7 marks]

(c) Let \( \rightarrow \) be a binary relation between pairs of positive natural numbers for which

\[
(m, n) \rightarrow (m, n - m) \text{ if } m < n, \quad \text{and} \quad (m, n) \rightarrow (m - n, n) \text{ if } n < m .
\]

Using (a) and (b), or otherwise, show that for all pairs of positive natural numbers \( (m, n) \), there is a natural number \( h \) such that

\[
(m, n) \rightarrow^* (h, h) .
\]

[8 marks]