6 Numerical Analysis (abr28)

(a) You are given a system of real equations in matrix form $Ax = b$ where $A$ is non-singular. Give three factorization techniques to solve this system, depending on the shape and structure of $A$: tall, square, symmetric. For each technique, give the relevant matrix equations to obtain the solution $x$, and point out the properties of the matrices involved. Highlight one potential problem from an implementation (computer representation) standpoint. [Note: You do not need to detail the factorization steps that give the matrix entries.] [5 marks]

(b) We want to estimate travel times between stops in a bus network, using ticketing data. The network is represented as a directed graph, with a vertex for each bus stop, and edges between adjacent stops along a route. For each edge $j \in \{1, \ldots, p\}$ let the travel time be $d_j$. The following ticketing data is available: for each trip $i \in \{1, \ldots, n\}$, we know its start time $s_i$, its end time $f_i$, and also the list of edges it traverses. The total trip duration is the sum of travel times along its edges.

We shall estimate the $d_j$ using linear least squares estimation, i.e. solve $\arg \min_\beta \|y - X\beta\|^2$ for a suitable matrix $X$ and vectors $\beta$ and $y$.

(i) Give an example of ticket data for a trip traversing 5 edges, and write the corresponding equation of its residual. [1 mark]

(ii) Give the dimensions and contents of $X$, $\beta$, and $y$ for this problem. State a condition on $X$ that ensures we can solve for $\beta$. [3 marks]

(iii) Give an example with $p = 2$ and $n = 3$ for which it is not possible to estimate the $d_j$. Compute $X^T X$ for your example. [2 marks]

(c) Let $A$ be an $n \times n$ matrix with real entries.

(i) We say that $A$ is diagonalisable if there exists an invertible $n \times n$ matrix $P$ such that the matrix $D = P^{-1} AP$ is diagonal. Show that if $A$ is diagonalisable and has only one eigenvalue then $A$ is a constant multiple of the identity matrix. [3 marks]

(ii) Let $A$ be such that when acting on vectors $x = [x_1, x_2, \ldots, x_n]^T$ it gives $Ax = [x_1, x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n]^T$. Write out the contents of $A$ and find its eigenvalues and eigenvectors. Scale the eigenvectors so they have unit length (i.e. so their magnitude is equal to 1). [6 marks]