1 Foundations of Computer Science (am21)

Three alternative representations for non-negative integers, \( n \), are:

- **Peano**: values have the form \( S(... S(Z)...) \), applying \( S \) \( n \) times to \( Z \) where \( S \) and \( Z \) are constructors or constants of some data type.

- **Binary**: values are of type \( \text{bool list} \) with 0 being represented as the empty list, and the least-significant bit being stored in the head of the list.

- **Church**: values have the form \( \text{fn f => fn x => f(... f(x) ...)} \), applying \( f \) \( n \) times to \( x \)

(a) Write ML functions for each of these data types which take the representation of an integer \( n \) as argument and return \( n \) as an ML \texttt{int}. [6 marks]

(b) Write ML functions for each of these data types which take representations of integers \( m \) and \( n \) and return the representation of \( m + n \). Your answers must not use any value or operation on type \texttt{int} or \texttt{real}. [Hint: you might it useful to write a function \texttt{majority: bool*bool*bool -> bool} (which returns true when two or more of its arguments are true) and to note that the ML inequality operator ‘\(<\)’ acts as exclusive-or on \texttt{bool}.] [10 marks]

(c) Letting \texttt{two} and \texttt{three} respectively be the Church representations of integers 2 and 3, indicate whether each of the following ML expressions give a Church representation of some integer and, if so what integer is represented, and if not giving a one-line reason.

\begin{itemize}
  \item[(i)] \texttt{two \ three}
  \item[(ii)] \texttt{three \ two}
  \item[(iii)] \texttt{two \circ \ three}
  \item[(iv)] \texttt{three \circ \ two}
\end{itemize} [4 marks]