1 Foundations of Computer Science (am21)

Three alternative representations for non-negative integers, \( n \), are:

- **Peano**: values have the form \( S(... S(Z) ...) \), applying \( S \) \( n \) times to \( Z \) where \( S \) and \( Z \) are constructors or constants of some data type.

- **Binary**: values are of type \texttt{bool list} with 0 being represented as the empty list, and the least-significant bit being stored in the head of the list.

- **Church**: values have the form \( fn f => fn x => f(... f(x) ...) \), applying \( f \) \( n \) times to \( x \)

(a) Write ML functions for each of these data types which take the representation of an integer \( n \) as argument and return \( n \) as an ML \texttt{int}. [6 marks]

(b) Write ML functions for each of these data types which take representations of integers \( m \) and \( n \) and return the representation of \( m + n \). Your answers must not use any value or operation on type \texttt{int} or \texttt{real}. [Hint: you might it useful to write a function \texttt{majority}: bool*bool*bool -> bool (which returns true when two or more of its arguments are true) and to note that the ML inequality operator ‘\(<\)’ acts as exclusive-or on \texttt{bool}.] [10 marks]

(c) Letting \texttt{two} and \texttt{three} respectively be the Church representations of integers 2 and 3, indicate whether each of the following ML expressions give a Church representation of some integer and, if so what integer is represented, and if not giving a one-line reason.

(i) \texttt{two three}

(ii) \texttt{three two}

(iii) \texttt{two o three}

(iv) \texttt{three o two} [4 marks]