5 Denotational Semantics (MPF)

(*a*) Give the definitions of poset (partially ordered set), cpo (chain complete poset), and domain. [2 marks]

(*b*) An *ideal* I of a poset $\mathsf{P} = (P, \leq)$ is a subset I of the set P such that:

- I is non-empty,
- for every $x \in I$ and $y \in P$, $y \leq x$ implies $y \in I$, and
- for every $x, y \in I$, there is $u \in I$ such that $x \leq u$ and $y \leq u$.

We write $\mathit{Idl}(\mathsf{P})$, where P is a poset, for the set of all the ideals of P.

(*i*) Show that $(\mathit{Idl}(\mathsf{P}), \subseteq)$, where $\subseteq$ denotes the subset-inclusion relation, is a cpo. [3 marks]

(*ii*) Show that $\mathit{Idl}(P, \leq)$ is non-empty iff P is non-empty. [2 marks]

(*iii*) Give a necessary and sufficient condition on a poset P for the cpo $(\mathit{Idl}(\mathsf{P}), \subseteq)$ to be a domain. Justify your answer. [2 marks]

(*c*) Give the definitions of monotone function between posets and of continuous function between cpos. [2 marks]

(*d*) (*i*) For a monotone function $f : \mathsf{P} \to \mathsf{Q}$ between posets P and Q, define a continuous function $f^{\#} : (\mathit{Idl}(\mathsf{P}), \subseteq) \to (\mathit{Idl}(\mathsf{Q}), \subseteq)$ between the cpos $(\mathit{Idl}(\mathsf{P}), \subseteq)$ and $(\mathit{Idl}(\mathsf{Q}), \subseteq)$. Prove that your definition is as requested. [4 marks]

(*ii*) For the identity function id_{P} on a poset P, show that $(id_{\mathsf{P}})^{\#}$ is the identity function on the cpo $(\mathit{Idl}(\mathsf{P}), \subseteq)$. [1 mark]

(*iii*) For monotone functions $f : \mathsf{P} \to \mathsf{Q}$ and $g : \mathsf{O} \to \mathsf{P}$ between posets O, P, and Q, show that $f^{\#} \circ g^{\#} = (f \circ g)^{\#} : (\mathit{Idl}(\mathsf{O}), \subseteq) \to (\mathit{Idl}(\mathsf{Q}), \subseteq)$. [4 marks]