8 Semantics of Programming Languages (PMS)

Consider the following syntax up to alpha equivalence, where n ranges over natural numbers, x over a set of variables, and (as usual) x is binding in e in $\mathbf{fn}\, x \Rightarrow e$.

expressions, $e ::= \quad n \mid x \mid \mathbf{fn}\, x \Rightarrow e \mid e\ e'$
values, $v ::= \quad n \mid x \mid \mathbf{fn}\, x \Rightarrow e$

(*a*) Define free variables $\mathrm{fv}(e)$ and capture-avoiding substitution $\{e/z\}e'$. [3 marks]

(*b*) Define a left-to-right call-by-value reduction relation $e \longrightarrow e'$. [3 marks]

Implementing a language using substitution is inefficient, as each substitution has to traverse a potentially large subterm. Consider the following proposal for an abstract machine for this language using environments E, lists of variable/value pairs.

$\boxed{\langle E, e\rangle \longrightarrow \langle E', e'\rangle}$

$$\frac{(x, v) \in E}{\langle E, x\rangle \longrightarrow \langle E, v\rangle}\ \textsc{lookup}$$

$$\frac{x \notin \mathrm{dom}(E) \cup \mathrm{fv}(\mathrm{range}(E)) \cup \mathrm{fv}(v)}{\langle E, (\mathbf{fn}\, x \Rightarrow e)\, v\rangle \longrightarrow \langle (x, v) :: E, e\rangle}\ \textsc{fn}$$

$$\frac{\langle E, e_1\rangle \longrightarrow \langle E', e_1'\rangle}{\langle E, e_1\ e_2\rangle \longrightarrow \langle E', e_1'\ e_2\rangle}\ \textsc{app\_left}$$

$$\frac{\langle E, e_2\rangle \longrightarrow \langle E', e_2'\rangle}{\langle E, v_1\ e_2\rangle \longrightarrow \langle E', v_1\ e_2'\rangle}\ \textsc{app\_right}$$

(*c*) Give the sequence of abstract-machine reduction steps, including the configurations and the names of the rules used, for the initial configuration below. You need not give full derivation trees.

$$\langle [], ((\mathbf{fn}\, x \Rightarrow (\mathbf{fn}\, y \Rightarrow x\ y))\, (\mathbf{fn}\, z \Rightarrow z))\, 3\rangle$$

[5 marks]

(*d*) Explain, with a concrete example and its reduction sequence, what could go wrong if the premise of FN had been omitted. [5 marks]

(*e*) Write $\{E\}e$ for the iterated substitution defined by

$$\begin{array}{lcl} \{[]\}e & = & e \\ \{(x, v) :: E\}e & = & \{E\}(\{v/x\}e) \end{array}$$

Prove that $\{E\}(e_1\ e_2) = (\{E\}e_1\ \{E\}e_2)$. [4 marks]