## 9 Discrete Mathematics (MPF)

(*a*) Define $F_0 = 0$, $F_1 = 1$ and for $n \in \mathbb{N}$, $F_{n+2} = F_{n+1} + F_n$.

For positive integers $a$ and $b$, prove that

$$\forall\, n \in \mathbb{N}.\ \gcd(aF_{n+3} + bF_{n+2},\, aF_{n+1} + bF_n) = \gcd(a, b)$$ [7 marks]

(*b*) Let $U$ be a set and let $\mathcal{P}(U)$ denote its powerset.

For $\mathcal{F} \subseteq \mathcal{P}(U)$, define $\mathcal{G} \subseteq \mathcal{P}(U)$ as $\{X \subseteq U \mid \forall\, S \in \mathcal{F}.\ S \subseteq X\}$.

Prove that $\bigcup \mathcal{F} = \bigcap \mathcal{G}$. [4 marks]

(*c*) For $i = 0, 1$, let $M_i = (Q_i, \Sigma, \delta_i, s_i, F_i)$ be deterministic finite automata, where $\delta_i : Q_i \times \Sigma \to Q_i$ are the next-state functions.

A relation $R \subseteq Q_0 \times Q_1$ is said to be a *simulation* whenever

$$\begin{array}{l} \forall\, q \in Q_0, q' \in Q_1. \\ \quad q\, R\, q' \implies \big[\ (q \in F_0 \Rightarrow q' \in F_1) \wedge \forall\, a \in \Sigma.\ \delta_0(q, a)\, R\, \delta_1(q', a)\ \big] \end{array}$$

(*i*) For $i = 0, 1$, let $\delta_i^{\#} : Q_i \times \Sigma^* \to Q_i$ be defined, for $q \in Q_i$, $a \in \Sigma$ and $w \in \Sigma^*$, by

$$\begin{aligned} \delta_i^{\#}(q, \varepsilon) &= q \\ \delta_i^{\#}(q, aw) &= \delta_i^{\#}(\delta_i(q, a), w) \end{aligned}$$

For a simulation $R \subseteq Q_0 \times Q_1$, prove that

$$\forall\, w \in \Sigma^*.\ \forall\, q \in Q_0, q' \in Q_1.\ q\, R\, q' \implies \delta_0^{\#}(q, w)\, R\, \delta_1^{\#}(q', w)$$ [7 marks]

(*ii*) For $i = 0, 1$, define $L(M_i) = \{w \in \Sigma^* \mid \delta_i^{\#}(s_i, w) \in F_i\}$.

Prove that if there exists a simulation $R \subseteq Q_0 \times Q_1$ such that $s_0\, R\, s_1$ then $L(M_0) \subseteq L(M_1)$. [2 marks]