10 Discrete Mathematics (IML)

(a) Let $\Sigma = \{a, b\}$ and let $\#_x(w)$ be the number of occurrences of the symbol $x \in \Sigma$ in the string $w$. For each of the following determine, with justification, whether or not the language is regular.

(i) $L_1 = \{w \in \Sigma^* \mid \#_a(w) = \#_b(w)\}$ [3 marks]

(ii) $L_2 = \{w \in \Sigma^* \mid w$ has an equal number of occurrences of the substrings $ab$ and $ba\}$. [3 marks]

(iii) $L_3$ inductively defined by the following axiom and rule:

\[
\begin{align*}
\epsilon & \quad \frac{u}{a} \quad \frac{b}{u} \quad \text{for all } u \in \Sigma^* \\
\end{align*}
\] [3 marks]

(iv) $L_4$ inductively defined by the following axiom and rules:

\[
\begin{align*}
\epsilon & \quad \frac{u}{a} \quad \frac{b}{u} \quad \frac{u}{a} \quad \frac{b}{u} \quad \text{for all } u \in \Sigma^* \\
\end{align*}
\] [3 marks]

(v) $L_5 = \{w \in \Sigma^* \mid (#_a(w) = 3i) \land (#_b(w) = 7j) \text{ for some } i, j \in \mathbb{N}\}$ [3 marks]

(b) Consider the set $R$ of all regular expressions over the alphabet $\{a, b\}$.

(i) Give an alphabet sufficient to express any element of $R$. [2 marks]

(ii) State, giving reasons, whether $R$ is a regular language. [3 marks]