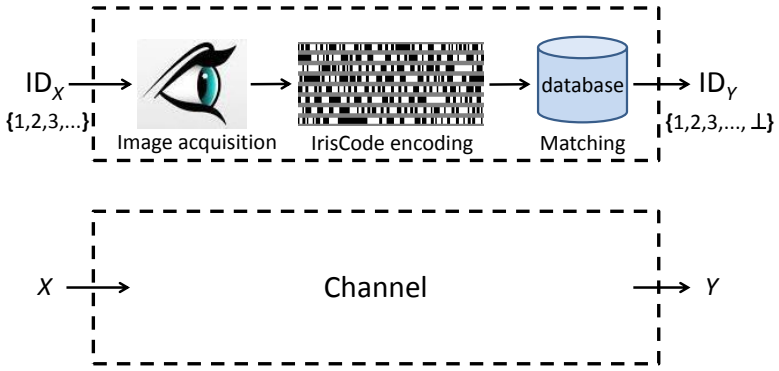
9 Information Theory (JGD)

A classifier or pattern recognition system such as illustrated in the diagram below can be treated abstractly as a channel, with some object as input represented by discrete random variable $X = \{x_1, \ldots, x_J\}$, and the decision output represented by discrete random variable $Y = \{y_1, \ldots, y_K\}$. Note that J and K need not be the same. Errors may be made, and the output $\perp$ may signify that no decision was possible.



(*a*) Define a channel matrix and provide it for such a system. [4 marks]

(*b*) Assuming there is a known probability distribution $p(x_j)$ over all possible inputs x_j, and that correct decisions about them are y_k with $k = j$, give an expression for the average probability of error, P_e. [4 marks]

(*c*) Give an expression for the mutual information $I(X;Y)$ of this system, in terms of the entropy $H(X)$ of the set of possible inputs and the conditional entropy $H(X|Y)$ of the inputs given the output decisions Y. [2 marks]

(*d*) Now give an expression for the mutual information $I(Y;X)$ of this system in terms of the entropy $H(Y)$ of the output decisions and the conditional entropy $H(Y|X)$ of the decisions given the inputs X. [2 marks]

(*e*) Suppose now that there are $J = 2^N$ (for integer N) possible inputs and that they are all equiprobable: $\forall j,\ p(x_j) = J^{-1} = 2^{-N}$. In terms of N, what is the entropy $H(X)$ of this set of possible inputs? [3 marks]

(*f*) In India, $J \approx 1$ billion $\approx 2^{30}$ citizens have been enrolled in the identification system illustrated in the diagram. All citizens present themselves equiprobably. Suppose that the average remaining uncertainty about input persons X given inferred identities Y is $H(X|Y) = 1$ bit: the system computes that the IrisCode may equally well arise from either one of two persons among the billion. What then is the mutual information of the system, in bits per identity? [3 marks]

(*g*) What then is the channel capacity C of this system, in bits per identity, given the assumption of equiprobable presenting identities? [2 marks]