5 Numerical Methods (DJG)

(a) Consider implementing the natural logarithm function $\ln(t)$ for floating-point numbers using the McLaurin series:

$$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

(i) List all special behaviours the natural logarithm function should have in different parts of its range and when $t$ takes the special values NaN and $\pm\infty$. [2 marks]

(ii) The function must accept a broad range of numerical values but the series only converges when the absolute value of $x$ is less than one, $|x| < 1$. Describe a range-reduction procedure that pre-processes the argument and post-processes the result so that the series always acts on small values of $x$. [6 marks]

(iii) State the two precision requirements normally expected for mathematical libraries. Considering the worst-case value(s) of $x$ after range reduction, approximately how many terms are needed to meet one of these requirements for a single-precision implementation? Do you expect the other requirement to be met? [6 marks]
(b) The Trapezoidal Rule for numerical definite integration returns the area of the trapezium-shaped strips formed by each pair of adjacent points. The area under each such strip is:

\[ \int_{a}^{b} f(x) \, dx \approx \frac{b-a}{2} [f(a) + f(b)] \]

(i) A program computes the area between two points \( A \) and \( B \) using \( N \) strips of width \( h \). What should be taken into account when choosing \( h \)? Suggest a good value for \( h \). [3 marks]

(ii) Assuming the best choice for \( h \), what characteristics of \( f() \) will affect the accuracy achieved? [3 marks]