9 Discrete Mathematics (MPF)

(a) Let \( r \) and \( s \) be solutions to the quadratic equation \( x^2 - bx + c = 0 \).

For \( n \in \mathbb{N} \), define
\[
\begin{align*}
  d_0 &= 0 \\
  d_1 &= r - s \\
  d_n &= b d_{n-1} - c d_{n-2} \quad (n \geq 2)
\end{align*}
\]
Prove that \( d_n = r^n - s^n \) for all \( n \in \mathbb{N} \). [4 marks]

(b) Recall that a commutative monoid is a structure \((M, 1, *)\) where \( M \) is a set, \( 1 \) is an element of \( M \), and \( * \) is a binary operation on \( M \) such that
\[
\begin{align*}
  x * 1 &= x \\
  x * y &= y * x \\
  (x * y) * z &= x * (y * z)
\end{align*}
\]
for all \( x, y, z \) in \( M \).

For a commutative monoid \((M, 1, *)\), consider the structure \((\mathcal{P}(M), I, \oplus)\) where \( \mathcal{P}(M) \) is the powerset of \( M \), \( I \) in \( \mathcal{P}(M) \) is the singleton set \( \{1\} \), and \( \oplus \) is the binary operation on \( \mathcal{P}(M) \) given by
\[
X \oplus Y = \{ m \in M \mid \exists x \in X. \exists y \in Y. m = x * y \}
\]
for all \( X \) and \( Y \) in \( \mathcal{P}(M) \).

Prove that \((\mathcal{P}(M), I, \oplus)\) is a commutative monoid. [10 marks]

(c) Define a section-retraction pair to be a pair of functions \((s : A \to B, r : B \to A)\) such that \( r \circ s = \text{id}_A \).

(i) Prove that for every section-retraction pair \((s, r)\), the section \( s \) is injective and the retraction \( r \) is surjective. [4 marks]

(ii) Exhibit two sets \( A \) and \( B \) together with an injective function \( f : A \to B \) such that there is no function \( g : B \to A \) for which \((f, g)\) is a section-retraction pair. [2 marks]