8 Discrete Mathematics (MPF)

(*a*) For a non-empty tuple of positive integers $a_1, \ldots, a_n$, let

$$\mathrm{CD}(a_1, ..., a_n) \;=\; \{\, d \in \mathbb{N} \,:\, \forall\, 1 \leq i \leq n.\ d \mid a_i \,\}$$

be the set of natural numbers that are common divisors of all $a_1, \ldots, a_n$.

(*i*) Without using the Fundamental Theorem of Arithmetic, prove that for positive integers a and a', if $\mathrm{CD}(a, a') = \{1\}$ then, for all integers k,

$$(a \cdot a') \mid k \iff a \mid k \,\wedge\, a' \mid k$$

[4 marks]

(*ii*) Either prove or disprove that, for all natural numbers $n \geq 2$ and all tuples of positive integers $a_1, \ldots, a_n$, if $\mathrm{CD}(a_1, \ldots, a_n) = \{1\}$ then, for all integers k, $(a_1 \cdot \ldots \cdot a_n) \mid k \implies a_1 \mid k \wedge \ldots \wedge a_n \mid k$. [3 marks]

(*iii*) Either prove or disprove that, for all natural numbers $n \geq 2$ and all tuples of positive integers $a_1, \ldots, a_n$, if $\mathrm{CD}(a_1, \ldots, a_n) = \{1\}$ then, for all integers k, $a_1 \mid k \wedge \ldots \wedge a_n \mid k \implies (a_1 \cdot \ldots \cdot a_n) \mid k$. [3 marks]

(*b*) Either prove or disprove that for all sets A, B, X, Y,

$$(\, A \cong X \,\wedge\, B \cong Y \,) \implies A \times B \,\cong\, Y \times X$$

[4 marks]

(*c*) (*i*) Define the notion of a surjective function between two sets. [2 marks]

(*ii*) State whether or not the function $f : \mathbb{N} \to \{n \in \mathbb{N} \mid n \geq 1\}$ defined by

$$f(0) = 1$$

$$f(n+1) = \begin{cases} f(n)/2 & \text{if } f(n) \text{ is even} \\ 9 \cdot f(n) + 1 & \text{otherwise} \end{cases} \qquad (n \in \mathbb{N})$$

is surjective. Prove your claim. [4 marks]