7 Discrete Mathematics (MPF)

(*a*) (*i*) Calculate $\gcd(144, 77)$, the greatest common divisor of 144 and 77, as an integer linear combination of 144 and 77. [4 marks]

(*ii*) What is the multiplicative inverse of 77 in $\mathbb{Z}_{144}$ and the multiplicative inverse of 67 in $\mathbb{Z}_{77}$? [2 marks]

(*iii*) Describe all integers x that solve the following two congruences

$$\begin{cases} 77 \cdot x \equiv 1 \pmod{144} \\ 67 \cdot x \equiv 3 \pmod{77} \end{cases}$$

Indicate how one may calculate the least natural number solution to the above. [4 marks]

Justify your answers.

(*b*) For a string $w \in \{1, 2\}^*$, let $\sum(w) \in \mathbb{N}$ denote the sum of all the numbers in it. For instance, $\sum(\varepsilon) = 0$ for ε the null string, and $\sum(1212) = 6$.

For every $n \in \mathbb{N}$, define $S_n = \{ w \in \{1, 2\}^* \mid \sum(w) = n \}$. In particular, $\varepsilon \in S_0$ and $1212 \in S_6$.

(*i*) List the elements of S_n for each $n \in \{0, 1, 2, 3, 4, 5\}$. [2 marks]

(*ii*) What is the cardinality of S_n for each $n \in \mathbb{N}$? Prove your claim. [5 marks]

(*iii*) For all $m, n \in \mathbb{N}$, define a bijective function

$$\big((S_{m+1} \times S_{n+1}) \uplus (S_m \times S_n)\big) \to S_{m+n+2}$$ [3 marks]