4 Computer Systems Modelling (RJG)

(a) (i) Suppose that $F_X(x)$ is a distribution function. Show the inverse transform result, namely that, if $U$ is a random variable uniformly distributed in the interval $(0, 1)$ then

$$X = F_X^{-1}(U)$$

is a random variable with distribution function $P(X \leq x) = F_X(x)$. [4 marks]

(ii) Discuss the notion of a pseudo-random number generator for uniform random variables. Describe suitable algorithms for generating pseudo-random numbers. [6 marks]

(iii) Using the inverse transform result in part (a)(i) derive a method to generate a stream of independent pseudo-random numbers from an exponential distribution with parameter $\lambda > 0$. What are the true mean and variance of these numbers in terms of $\lambda$? [4 marks]

(b) (i) Suppose that you conduct a simulation experiment to estimate the mean, $\mu$, of a random quantity $X$ from a sample of $n$ values $X_1, X_2, \ldots, X_n$. How would you estimate $\mu$? [2 marks]

(ii) Now suppose that your simulation also yields a sample of $n$ values $Y_1, Y_2, \ldots, Y_n$ of the random quantity $Y$ where $E(Y) = \mu_Y$ is a known number. How would you use the method of control variates to improve your estimator of $\mu$? Your answer should mention all quantities that may need to be estimated and in what way you will improve the estimation of $\mu$. [4 marks]