7 Denotational Semantics (MPF)

(*a*) (*i*) Define the notion of continuous function between domains. [2 marks]

(*ii*) Let $\mathcal{P}(\mathbb{N}^2)$ be the domain of all subsets of pairs of natural numbers ordered by inclusion. Show that the function $f : \mathcal{P}(\mathbb{N}^2) \to \mathcal{P}(\mathbb{N}^2)$ given by

$$f(S) = \{ (1,1) \} \cup \{ (x+1, x \cdot y) \in \mathbb{N}^2 \mid (x,y) \in S \} \qquad (S \subseteq \mathbb{N}^2)$$

is continuous. [3 marks]

(*b*) (*i*) State Tarski's fixed point theorem for a continuous endofunction on a domain. [2 marks]

(*ii*) Give a concrete explicit description of the fixed point $\mathit{fix}(f) \subseteq \mathbb{N}^2$ of the continuous function f in Part (*a*)(*ii*). Briefly justify your answer. [3 marks]

(*c*) (*i*) Define the notion of an admissible subset of a domain. [2 marks]

(*ii*) Let $P \subseteq \mathcal{P}(\mathbb{N}^2)$ be defined as $P = \{ S \subseteq \mathbb{N}^2 \mid \forall (x,y) \in S.\ \log y \leq x \cdot \log x \}$. Show that P is an admissible subset of the domain $\mathcal{P}(\mathbb{N}^2)$. [3 marks]

(*d*) (*i*) State Scott's fixed point induction principle. [2 marks]

(*ii*) Use Scott's fixed point induction principle to show that $\mathit{fix}(f) \in P$ for f the continuous function in Part (*a*)(*ii*) and P the admissible subset of the domain $\mathcal{P}(\mathbb{N}^2)$ in Part (*c*)(*ii*). [3 marks]