7 Discrete Mathematics (MPF)

You may use standard results provided that you mention them clearly.

(*a*) (*i*) State a sufficient condition on a pair of positive integers a and b so that the following holds:

$$\forall\, x, y \in \mathbb{Z}.\ \big(x \equiv y \pmod{a} \wedge x \equiv y \pmod{b}\big) \iff x \equiv y \pmod{ab}$$

[2 marks]

(*ii*) Recall that, for a positive integer m, we let $\mathbb{Z}_m = \{n \in \mathbb{N} \mid 0 \leq n < m\}$ and that, for an integer k, we write $[k]_m$ for the unique element of $\mathbb{Z}_m$ such that $k \equiv [k]_m \pmod{m}$.

Let a and b be positive integers and let k and l be integers such that $k\,a + l\,b = 1$. Consider the functions $f : \mathbb{Z}_{ab} \to \mathbb{Z}_a \times \mathbb{Z}_b$ and $g : \mathbb{Z}_a \times \mathbb{Z}_b \to \mathbb{Z}_{ab}$ given by

$$f(n) = \big(\, [n]_a\, ,\ [n]_b\, \big) \quad , \quad g(x, y) = [k\, a\, (y - x) + x]_{ab}$$

Prove either that $g \circ f = \mathrm{id}_{\mathbb{Z}_{ab}}$ or that $f \circ g = \mathrm{id}_{\mathbb{Z}_a \times \mathbb{Z}_b}$. [8 marks]

(*b*) Let T^* denote the reflexive-transitive closure of a relation T on a set A.

For relations R and S on a set A, prove that if $\mathrm{id}_A \subseteq (R \cap S)$ then $(R \cup S)^* = (R \circ S)^*$.

Note: You may alternatively consider T^* to be defined as either

$$\bigcup_{n \in \mathbb{N}} T^{\circ n} \quad \text{, where } T^{\circ 0} = \mathrm{id}_A \text{ and } T^{\circ (n+1)} = T \circ T^{\circ n}$$

or as

$$\bigcap \{\, R \subseteq A \times A \mid (T \cup \mathrm{id}_A) \subseteq R \ \wedge\ R \circ R \subseteq R \,\}$$

or as inductively given by the rules

$$\frac{}{(x, y)}\ \big((x, y) \in T\big) \qquad \frac{}{(x, x)}\ (x \in A) \qquad \frac{(x, y)\ \ (y, z)}{(x, z)}\ (x, y, z \in A)$$

[10 marks]