7 Discrete Mathematics (MPF)

You may use standard results provided that you mention them clearly.

(a) (i) State a sufficient condition on a pair of positive integers \(a\) and \(b\) so that the following holds:

\[
\forall x, y \in \mathbb{Z}. \ (x \equiv y \pmod{a} \land x \equiv y \pmod{b}) \iff x \equiv y \pmod{ab}
\]

[2 marks]

(ii) Recall that, for a positive integer \(m\), we let \(\mathbb{Z}_m = \{n \in \mathbb{N} \mid 0 \leq n < m\}\) and that, for an integer \(k\), we write \([k]_m\) for the unique element of \(\mathbb{Z}_m\) such that \(k \equiv [k]_m \pmod{m}\).

Let \(a\) and \(b\) be positive integers and let \(k\) and \(l\) be integers such that \(ka + lb = 1\). Consider the functions \(f : \mathbb{Z}_{ab} \to \mathbb{Z}_a \times \mathbb{Z}_b\) and \(g : \mathbb{Z}_a \times \mathbb{Z}_b \to \mathbb{Z}_{ab}\) given by

\[
f(n) = ([n]_a, [n]_b), \quad g(x, y) = [ka(y - x) + x]_{ab}
\]

Prove either that \(g \circ f = \text{id}_{\mathbb{Z}_{ab}}\) or that \(f \circ g = \text{id}_{\mathbb{Z}_a \times \mathbb{Z}_b}\).

[8 marks]

(b) Let \(T^*\) denote the reflexive-transitive closure of a relation \(T\) on a set \(A\).

For relations \(R\) and \(S\) on a set \(A\), prove that if \(\text{id}_A \subseteq (R \cap S)\) then \((R \cup S)^* = (R \circ S)^*\).

Note: You may alternatively consider \(T^*\) to be defined as either

\[
\bigcup_{n \in \mathbb{N}} T^{\circ n}, \text{ where } T^{\circ 0} = \text{id}_A \text{ and } T^{\circ (n+1)} = T \circ T^{\circ n}
\]

or as

\[
\bigcap \{ R \subseteq A \times A \mid (T \cup \text{id}_A) \subseteq R \land R \circ R \subseteq R \}
\]

or as inductively given by the rules

\[
\frac{(x, y) \in T}{(x, y)} \quad \frac{(x \in A)}{(x, x)} \quad \frac{(x, y), (y, z) \in A}{(x, z)} \quad (x, y, z \in A)
\]

[10 marks]