7 Discrete Mathematics (MPF)

You may use standard results provided that you mention them clearly.

(a) (i) State a sufficient condition on a pair of positive integers $a$ and $b$ so that the following holds:

$$\forall x, y \in \mathbb{Z}. (x \equiv y \pmod{a} \land x \equiv y \pmod{b}) \iff x \equiv y \pmod{ab}$$

[2 marks]

(ii) Recall that, for a positive integer $m$, we let $\mathbb{Z}_m = \{ n \in \mathbb{N} \mid 0 \leq n < m \}$ and that, for an integer $k$, we write $[k]_m$ for the unique element of $\mathbb{Z}_m$ such that $k \equiv [k]_m \pmod{m}$.

Let $a$ and $b$ be positive integers and let $k$ and $l$ be integers such that $ka + lb = 1$. Consider the functions $f : \mathbb{Z}_{ab} \to \mathbb{Z}_a \times \mathbb{Z}_b$ and $g : \mathbb{Z}_a \times \mathbb{Z}_b \to \mathbb{Z}_{ab}$ given by

$$f(n) = ([n]_a, [n]_b), \quad g(x, y) = [ka(y - x) + x]_{ab}$$

Prove either that $g \circ f = \text{id}_{\mathbb{Z}_{ab}}$ or that $f \circ g = \text{id}_{\mathbb{Z}_a \times \mathbb{Z}_b}$. [8 marks]

(b) Let $T^*$ denote the reflexive-transitive closure of a relation $T$ on a set $A$.

For relations $R$ and $S$ on a set $A$, prove that if $\text{id}_A \subseteq (R \cap S)$ then $(R \cup S)^* = (R \circ S)^*$.

Note: You may alternatively consider $T^*$ to be defined as either

$$\bigcup_{n \in \mathbb{N}} T^{\circ n}, \text{ where } T^{\circ 0} = \text{id}_A \text{ and } T^{\circ (n+1)} = T \circ T^{\circ n}$$

or as

$$\bigcap \{ R \subseteq A \times A \mid (T \cup \text{id}_A) \subseteq R \land R \circ R \subseteq R \}$$

or as inductively given by the rules

$$\frac{(x, y) (x, y) \in T}{(x, y)} \quad \frac{(x, x) (x \in A)}{(x, x)} \quad \frac{(x, y) (y, z) (x, y, z \in A)}{(x, z)}$$

[10 marks]