9 Information Theory and Coding (JGD)

(a) A two-state Markov process may emit a ‘0’ in State 0 or a ‘1’ in State 1, each with probability $\alpha$, and return to the same state; or with probability $1 - \alpha$ it emits the other symbol and switches to the other state. Thus it tends to be “sticky” or oscillatory, two forms of predictability, depending on $\alpha$.

(i) What are the state occupancy probabilities for $0 < \alpha < 1$? [2 marks]

(ii) What are the entropy of State 0, the entropy of State 1, and the overall entropy of this source? Express your answers in terms of $\alpha$. [2 marks]

(iii) For what value(s) of $\alpha$ do both forms of predictability disappear? What then is the entropy of this source, in bits per emitted bit? [2 marks]

(b) Consider a binary symmetric channel with error probability $p$ that any bit may be flipped. Two possible error-correcting coding schemes are available.

(i) Without any error-correcting coding scheme in place, state all the conditions that would maximise the channel capacity. Include conditions on the error probability $p$ and also on the probability distribution of the binary source input symbols. [2 marks]

(ii) If a (7/4) Hamming code is used to deliver error correction for up to one flipped bit in any block of seven bits, provide an expression for the residual error probability $P_e$ that such a scheme would fail. [2 marks]

(iii) If instead repetition were used to try to achieve error correction by repeating every message an odd number of times $N = 2m + 1$, for a positive integer $m$ followed by majority voting, provide an expression for the residual error probability $P_e$ that the repetition scheme would fail. [2 marks]

(c) Gabor wavelets are an important class of complex-valued functions for encoding information with maximal resolution simultaneously in the frequency domain and the signal domain. Using an expression for their functional form, explain:

(i) their spiral helical trajectory as phasors, shown here with projections of their real and imaginary parts;
(ii) the Uncertainty Principle under which they are optimal;
(iii) the spaces they occupy in the Information Diagram;
(iv) some of their uses in pattern encoding and recognition. [8 marks]