COMPUTER SCIENCE TRIPOS Part II – 2015 – Paper 7

6 Denotational Semantics (MPF)

- (a) For monotone functions $f, f': P \to Q$ between posets (P, \sqsubseteq_P) and (Q, \sqsubseteq_Q) , let $f \sqsubseteq f' \stackrel{\text{def}}{\iff} \forall x \in P. f(x) \sqsubseteq_Q f'(x).$
 - (i) Prove that the binary relation \sqsubseteq is a partial order. [3 marks]
 - (*ii*) For monotone functions between posets $p: P' \to P$, $f, f': P \to Q$, and $q: Q \to Q'$, prove that $f \sqsubseteq f' \implies q \circ f \circ p \sqsubseteq q \circ f' \circ p$. [1 mark]
- (b) An adjoint pair $(f : P \to Q, g : Q \to P)$ is a pair of monotone functions between posets such that $id_P \sqsubseteq g \circ f$ and $f \circ g \sqsubseteq id_Q$.
 - (i) Let $f_1, f_2: P \to Q$ and $g_1, g_2: Q \to P$ be monotone functions between posets such that (f_1, g_1) and (f_2, g_2) are adjoint pairs. Prove that:
 - (A) $f_1 \sqsubseteq f_2 \iff g_2 \sqsubseteq g_1$ [4 marks]

(B)
$$f_1 = f_2 \iff g_1 = g_2$$
 [2 marks]

(*ii*) Let $(f : P \to Q, g : Q \to P)$ be an adjoint pair where the posets P and Q have least elements. Prove that the monotone function f is strict.

[2 marks]

- (c) (i) Define the notion of lub (least upper bound) of a countable increasing chain in a poset. [2 marks]
 - (*ii*) Let $(f: D \to E, g: E \to D)$ be an adjoint pair where each of the posets D and E is a cpo (chain complete poset). Prove that the monotone function f is continuous. [6 marks]