2 Advanced Algorithms (TMS)

(a) State the fundamental theorem of linear programming. [3 marks]

(b) Consider the following linear program:

Minimize \(-3x_1 - 2x_2\)

subject to:

\[3x_1 + x_2 \leq 5\]
\[-2x_1 \geq -10 + 4x_2\]
\[x_1, x_2 \geq 0.\]

(i) Convert this LP into standard and slack form, and specify the initial basic solution. [4 marks]

(ii) Solve this LP using the simplex algorithm. Specify the associated basic solution after each iteration. [4 marks]

(c) We consider the Steiner Tree Problem defined as follows. We are given an undirected, connected graph \(G = (V, E)\) with a non-negative cost-function \(c : E \to \mathbb{R}_+\). Further, we are given a set \(S \subseteq V\) of terminals. The goal is to find a minimum-cost subgraph of \(G\) that connects all terminals, where the cost of a subgraph is the sum of the costs of its edges.

Consider the following algorithm:

- Let \(H = (V, E')\) be the metric completion of \(G\), where \(E' = \{(u, v) : u, v \in V\}\) and \(c(\{u, v\})\) is the cost of the shortest path from \(u\) to \(v\) in \(G\).
- Compute a Minimum Spanning Tree \(T\) on the subgraph \(H[S]\) induced by the set of terminals \(S\).
- Replace every edge \(\{u, v\}\) in \(T\) by the edges of a shortest path from \(u\) to \(v\) in \(G\), and return the solution.

(i) Prove an upper bound of \(2\left(1 - \frac{1}{|S|}\right)\) on the approximation ratio of this algorithm.

[Hint: You can use an approach similar to the analysis of APPROX-TSP-TOUR.] [6 marks]

(ii) Construct an example which provides a matching lower bound on the approximation ratio. [3 marks]