8 Mathematical Methods for Computer Science (RJG)

(a) Let $X$ be a random variable with finite mean $\mu = \mathbb{E}(X)$ and finite variance $\sigma^2 = \text{Var}(X)$. State and prove Chebyshev’s inequality for the random variable $X$. You may assume Markov’s inequality without proof. 

(b) Now suppose that $X$ is a continuous random variable with probability density function $f_X(x)$ and finite mean $\mu = \mathbb{E}(X)$ such that

- $f_X(x) = 0 \quad \forall x \notin [\alpha, \beta]$
- $xf_X(x) \leq \gamma \quad \forall x \in [\alpha, \beta]$

where $\alpha, \beta$ and $\gamma$ are non-negative real constants with $\alpha < \beta$. Suppose that $(A_i, B_i)$ for $i = 1, 2, \ldots, n$ is a sequence of independent and identically distributed 2-dimensional random variables where $A_i$ and $B_i$ are independent with marginal distributions $A_i \sim U[\alpha, \beta]$ and $B_i \sim U[0, \gamma]$ for each $i = 1, 2, \ldots, n$.

(i) Define random variables $I_i$ for $i = 1, 2, \ldots, n$ such that

$$I_i = \begin{cases} 
1 & \text{if } B_i \leq A_i f_X(A_i) \\
0 & \text{otherwise}
\end{cases}$$

and set $Z_n = \frac{1}{n} \sum_{i=1}^{n} I_i$. Show that $\mathbb{E}(Z_n) = \frac{\mu}{(\gamma(\beta - \alpha))}$ and that $\text{Var}(Z_n) \leq \frac{1}{4n}$. 

(ii) Using Chebyshev’s inequality show that $Z_n$ converges in probability to the degenerate random variable with value $\mu/(\gamma(\beta - \alpha))$. 

(iii) Describe an algorithm to estimate the mean $\mu$ of the random variable $X$. You may assume for the purpose of your algorithm that you have a function that returns random points of the given form $(A_i, B_i)$. 
