7 Mathematical Methods for Computer Science (JGD)

(a) An inner product space \( E \) containing piecewise continuous complex functions \( f(x) \) and \( g(x) \) on some interval is spanned by the orthonormal basis functions \( \{ e_i \} \) used in the Fourier series. Thus complex coefficients \( \{ \alpha_i \} \) and \( \{ \beta_i \} \) exist such that
\[
 f(x) = \sum_i \alpha_i e_i(x) \quad \text{and} \quad g(x) = \sum_i \beta_i e_i(x).
\]

(i) Show that \( \langle f, g \rangle = \sum_i \alpha_i \overline{\beta_i} \). [5 marks]

(ii) Would the same result hold if the orthonormal basis functions \( \{ e_i \} \) that span \( E \) were not the Fourier basis? Justify your answer, and provide the name for coefficients \( \{ \alpha_i \} \) and \( \{ \beta_i \} \) in such a case. [2 marks]

(b) Consider a sequence \( f[n] \) \((n = 0, 1, \ldots, 15)\) with Fourier coefficients \( F[k] \) \((k = 0, 1, \ldots, 15)\). Using the 16\(^{th}\) roots of unity as labelled around the unit circle in powers of \( w^1 \), the primitive 16\(^{th}\) root of unity, construct a sequence of these \( w^i \) that could be used to compute \( F[3] \). [4 marks]

(c) From the well-known fact that a periodic square wave \((f(x) = 1 \text{ for } 0 < x < \pi, \quad f(x) = -1 \text{ for } \pi < x < 2\pi, \ldots)\) has the following Fourier series
\[
 f(x) = \frac{4}{\pi} \left[ \sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7} + \cdots \right]
\]
produce the first four terms of the Fourier series for the triangle wave whose derivative is this square wave. [4 marks]

(d) What sets of frequencies are required to perform the following analyses?

- Fourier transform of a non-periodic continuous function
- Fourier analysis of a piecewise continuous periodic function with period \( 2\pi \)
- Wavelet transform of a non-periodic function, either continuous or discrete

Comment on the relationship between the density of frequencies required and the role of “locality” in the analysis. [5 marks]