

9 Discrete Mathematics (MPF)

- (a) Without using the Fundamental Theorem of Arithmetic, prove that for all positive integers  $a, b, c$ ,

$$\gcd(a, c) = 1 \implies (\gcd(a \cdot b, c) \mid b \wedge \gcd(a \cdot b, c) = \gcd(b, c))$$

You may use any other standard results provided that you state them clearly. [6 marks]

- (b) Prove that for all disjoint sets  $X$  and  $Y$  (that is, such that  $X \cap Y = \emptyset$ ),

$$\mathcal{P}(X \cup Y) \cong \mathcal{P}(X) \times \mathcal{P}(Y)$$

You may use any standard results provided that you state them clearly. [6 marks]

- (c) (i) For an alphabet  $\Sigma$  and a language  $L \subseteq \Sigma^*$ , define the language  $F(L) \subseteq \Sigma^*$  as

$$F(L) \stackrel{\text{def}}{=} \{awa \in \Sigma^* \mid a \in \Sigma \wedge w \in L\}$$

Prove that for all  $L_1, L_2 \subseteq \Sigma^*$ ,  $F(L_1 \cup L_2) = F(L_1) \cup F(L_2)$ . [2 marks]

- (ii) Let  $\text{Pal} \subseteq \Sigma^*$  be the language of palindromes (i.e. strings that read the same backwards as forwards) defined as  $\text{Pal} \stackrel{\text{def}}{=} \{w \in \Sigma^* \mid \text{rev}(w) = w\}$  where  $\text{rev}$  is the unique function  $\Sigma^* \rightarrow \Sigma^*$  such that  $\text{rev}(\varepsilon) = \varepsilon$ ,  $\text{rev}(a) = a$  for all  $a \in \Sigma$ , and  $\text{rev}(w_1w_2) = \text{rev}(w_2)\text{rev}(w_1)$  for all  $w_1, w_2 \in \Sigma^*$ .

Prove that for all  $L \subseteq \Sigma^*$ ,  $L \subseteq \text{Pal} \implies F(L) \subseteq \text{Pal}$ . [2 marks]

- (iii) For  $k \in \mathbb{N}$ , let  $\text{Pal}_k \subseteq \text{Pal}$  be the language of palindromes of length  $k$ ; that is,  $\text{Pal}_k \stackrel{\text{def}}{=} \{w \in \text{Pal} \mid |w| = k\}$ .

Recalling that  $F^0(X) \stackrel{\text{def}}{=} X$  and, for  $k \in \mathbb{N}$ ,  $F^{k+1}(X) \stackrel{\text{def}}{=} F(F^k(X))$ , prove that:

(A) for all  $n \in \mathbb{N}$ ,  $F^n(\text{Pal}_0) = \text{Pal}_{2n}$ , [2 marks]

(B) for all  $n \in \mathbb{N}$ ,  $F^n(\text{Pal}_1) = \text{Pal}_{2n+1}$ . [2 marks]

It follows that  $\bigcup_{n \in \mathbb{N}} F^n(\text{Pal}_0 \cup \text{Pal}_1) = \text{Pal}$ , though you are not required to prove this.