7 Discrete Mathematics (MPF)

(*a*) Let $\mathbb{N}_{\geq 2} \stackrel{\text{def}}{=} \{\, k \in \mathbb{N} \mid k \geq 2 \,\}$.

Without using the Fundamental Theorem of Arithmetic, prove that for all positive integers m and n,

$$\gcd(m, n) = 1 \iff \neg(\,\exists\, k \in \mathbb{N}_{\geq 2}.\ k \mid m \wedge k \mid n\,)$$

You may use any other standard results provided that you state them clearly. [6 marks]

(*b*) Recall that, for $i, j \in \mathbb{N}$,

$$\binom{i}{j} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{, if } i < j \\ \frac{i!}{j!(i-j)!} & \text{, if } i \geq j \end{cases}$$

(*i*) Show that for all $m < l$ in $\mathbb{N}$,

$$\binom{l}{m+1} + \binom{l}{m} = \binom{l+1}{m+1}$$

[2 marks]

(*ii*) Prove that

$$\forall\, n \in \mathbb{N}.\ \forall\, m \in \mathbb{N}.\ 0 \leq m \leq n \implies \sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

[6 marks]

(*c*) Let U be a set and let $F : \mathbb{N} \times \mathbb{N} \to \mathcal{P}(U)$ be a function such that for all $i, i', j, j' \in \mathbb{N}$, if $i \leq i'$ and $j \leq j'$ then $F(i, j) \subseteq F(i', j')$ in $\mathcal{P}(U)$.

Prove that

$$\bigcup_{i \in \mathbb{N}} \left(\bigcup_{j \in \mathbb{N}} F(i, j) \right) = \bigcup_{k \in \mathbb{N}} F(k, k)$$

(Recall that $x \in \bigcup_{l \in L} X_l \iff \exists\, l \in L.\ x \in X_l$.) [6 marks]