10 Discrete Mathematics (AMP)

(*a*) Give a *deterministic* finite automaton (DFA) with input alphabet $\{a\}$ accepting the language $\{a^n \mid n \in U\}$, where $U = \{1, 2\} \cup \{n \geq 3 \mid n \equiv 4 \pmod 6 \vee n \equiv 7 \pmod 6\}$. [3 marks]

(*b*) What does it mean for a language over an alphabet Σ to be *regular*? [2 marks]

(*c*) A subset U of the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers is called *ultimately periodic* if there exist numbers $N \geq 0$ and $p > 0$ such that for all $n \geq N$, $n \in U$ if and only if $n + p \in U$.

(*i*) Explain why every *finite* set of numbers is ultimately periodic according to the above definition. [2 marks]

(*ii*) Let L be a regular language over the alphabet $\{a\}$. By considering the shape of paths in the transition graph of any DFA with input alphabet $\{a\}$, or otherwise, show that $\{n \in \mathbb{N} \mid a^n \in L\}$ is an ultimately periodic set of numbers. [8 marks]

(*iii*) Conversely, show that if $U \subseteq \mathbb{N}$ is ultimately periodic, then $\{a^n \mid n \in U\}$ is a regular language. [5 marks]