10 Discrete Mathematics (AMP)

(a) Give a deterministic finite automaton (DFA) with input alphabet \{a\} accepting the language \{a^n \mid n \in U\}, where \( U = \{1, 2\} \cup \{n \geq 3 \mid n \equiv 4 \pmod{6} \vee n \equiv 7 \pmod{6}\} \). [3 marks]

(b) What does it mean for a language over an alphabet \( \Sigma \) to be regular? [2 marks]

(c) A subset \( U \) of the set \( \mathbb{N} = \{0, 1, 2, \ldots\} \) of natural numbers is called \textit{ultimately periodic} if there exist numbers \( N \geq 0 \) and \( p > 0 \) such that for all \( n \geq N \), \( n \in U \) if and only if \( n + p \in U \).

(i) Explain why every finite set of numbers is ultimately periodic according to the above definition. [2 marks]

(ii) Let \( L \) be a regular language over the alphabet \{a\}. By considering the shape of paths in the transition graph of any DFA with input alphabet \{a\}, or otherwise, show that \( \{n \in \mathbb{N} \mid a^n \in L\} \) is an ultimately periodic set of numbers. [8 marks]

(iii) Conversely, show that if \( U \subseteq \mathbb{N} \) is ultimately periodic, then \( \{a^n \mid n \in U\} \) is a regular language. [5 marks]