## 4 Computation Theory (AMP)

(*a*) Give the recursion equations for the function $\rho^n(f, g) \in \mathbb{N}^{n+1} \to \mathbb{N}$ *defined by primitive recursion* from functions $f \in \mathbb{N}^n \to \mathbb{N}$ and $g \in \mathbb{N}^{n+2} \to \mathbb{N}$. [2 marks]

(*b*) Define the class PRIM of *primitive recursive functions*, giving exact definitions for all the functions and operations you use. [5 marks]

(*c*) Show that the addition function $add(x, y) = x + y$ is in PRIM. [2 marks]

(*d*) Give an example of a function $\mathbb{N}^2 \to \mathbb{N}$ that is not in PRIM. [3 marks]

(*e*) The Fibonacci function $fib \in \mathbb{N} \to \mathbb{N}$ satisfies $fib(0) = 0$, $fib(1) = 1$ and $fib(x + 2) = fib(x) + fib(x + 1)$ for all $x \in \mathbb{N}$.

(*i*) Assuming the existence of primitive recursive functions $pair \in \mathbb{N}^2 \to \mathbb{N}$, $fst \in \mathbb{N} \to \mathbb{N}$ and $snd \in \mathbb{N} \to \mathbb{N}$ satisfying for all $x, y \in \mathbb{N}$

$$fst(pair(x, y)) = x \;\wedge\; snd(pair(x, y)) = y$$

prove by mathematical induction that any function $g \in \mathbb{N} \to \mathbb{N}$ satisfying

$$\begin{aligned} g(0) &= pair(0, 1) \\ g(x + 1) &= pair(snd(g(x)), fst(g(x)) + snd(g(x))) \end{aligned}$$

for all $x \in \mathbb{N}$, also satisfies

$$\forall x \in \mathbb{N}(fst(g(x)) = fib(x) \;\wedge\; snd(g(x)) = fib(x + 1)).$$

[4 marks]

(*ii*) Deduce that the Fibonacci function $fib$ is in PRIM. [4 marks]