4 Computation Theory (AMP)

(a) Give the recursion equations for the function \( \rho^n(f, g) \in \mathbb{N}^{n+1} \rightarrow \mathbb{N} \) defined by primitive recursion from functions \( f \in \mathbb{N}^n \rightarrow \mathbb{N} \) and \( g \in \mathbb{N}^{n+2} \rightarrow \mathbb{N} \). [2 marks]

(b) Define the class PRIM of primitive recursive functions, giving exact definitions for all the functions and operations you use. [5 marks]

(c) Show that the addition function \( \text{add}(x, y) = x + y \) is in PRIM. [2 marks]

(d) Give an example of a function \( \mathbb{N}^2 \rightarrow \mathbb{N} \) that is not in PRIM. [3 marks]

(e) The Fibonacci function \( \text{fib} \in \mathbb{N} \rightarrow \mathbb{N} \) satisfies \( \text{fib}(0) = 0 \), \( \text{fib}(1) = 1 \) and \( \text{fib}(x + 2) = \text{fib}(x) + \text{fib}(x + 1) \) for all \( x \in \mathbb{N} \).

   (i) Assuming the existence of primitive recursive functions \( \text{pair} \in \mathbb{N}^2 \rightarrow \mathbb{N} \), \( \text{fst} \in \mathbb{N} \rightarrow \mathbb{N} \) and \( \text{snd} \in \mathbb{N} \rightarrow \mathbb{N} \) satisfying for all \( x, y \in \mathbb{N} \)
   \[
   \text{fst}(\text{pair}(x, y)) = x \land \text{snd}(\text{pair}(x, y)) = y
   \]
   prove by mathematical induction that any function \( g \in \mathbb{N} \rightarrow \mathbb{N} \) satisfying
   \[
   g(0) = \text{pair}(0, 1)
   \]
   \[
   g(x + 1) = \text{pair}(\text{snd}(g(x)), \text{fst}(g(x)) + \text{snd}(g(x)))
   \]
   for all \( x \in \mathbb{N} \), also satisfies
   \[
   \forall x \in \mathbb{N}(\text{fst}(g(x)) = \text{fib}(x) \land \text{snd}(g(x)) = \text{fib}(x + 1)).
   \] [4 marks]

   (ii) Deduce that the Fibonacci function \( \text{fib} \) is in PRIM. [4 marks]