4 Computation Theory (AMP)

(a) Give the recursion equations for the function \( \rho^n(f, g) \in \mathbb{N}^{n+1} \rightarrow \mathbb{N} \) defined by primitive recursion from functions \( f \in \mathbb{N}^n \rightarrow \mathbb{N} \) and \( g \in \mathbb{N}^{n+2} \rightarrow \mathbb{N} \). [2 marks]

(b) Define the class PRIM of primitive recursive functions, giving exact definitions for all the functions and operations you use. [5 marks]

(c) Show that the addition function \( add(x, y) = x + y \) is in PRIM. [2 marks]

(d) Give an example of a function \( \mathbb{N}^2 \rightarrow \mathbb{N} \) that is not in PRIM. [3 marks]

(e) The Fibonacci function \( fib \in \mathbb{N} \rightarrow \mathbb{N} \) satisfies \( fib(0) = 0, fib(1) = 1 \) and \( fib(x + 2) = fib(x) + fib(x + 1) \) for all \( x \in \mathbb{N} \).

(i) Assuming the existence of primitive recursive functions \( pair \in \mathbb{N}^2 \rightarrow \mathbb{N}, \) \( fst \in \mathbb{N} \rightarrow \mathbb{N} \) and \( snd \in \mathbb{N} \rightarrow \mathbb{N} \) satisfying for all \( x, y \in \mathbb{N} \)

\[
\text{fst}(pair(x, y)) = x \land \text{snd}(pair(x, y)) = y
\]

prove by mathematical induction that any function \( g \in \mathbb{N} \rightarrow \mathbb{N} \) satisfying

\[
g(0) = pair(0, 1) \\
g(x + 1) = pair(snd(g(x)), fst(g(x)) + snd(g(x)))
\]

for all \( x \in \mathbb{N} \), also satisfies

\[
\forall x \in \mathbb{N}(\text{fst}(g(x)) = fib(x) \land \text{snd}(g(x)) = fib(x + 1)).
\]

[4 marks]

(ii) Deduce that the Fibonacci function \( fib \) is in PRIM. [4 marks]