4 Computation Theory (AMP)

(a) (i) What does it mean for a λ-term to be a β-normal form? Defining the sets of canonical (C) and neutral (U) λ-terms by the grammar

\[
C ::= \lambda x. C \mid U \\
U ::= x \mid UC
\]

show that a λ-term is a β-normal form if and only if it is canonical. [5 marks]

(ii) Carefully stating any standard properties of β-reduction, explain why a λ-term reduces to at most one β-normal form (up to α-equivalence). [4 marks]

(iii) Give an example of a λ-term that does not reduce to any β-normal form. [2 marks]

(b) (i) Define what it means for a closed λ-term \( F \) to represent a partial function \( f \in \mathbb{N} \rightarrow \mathbb{N} \). [4 marks]

(ii) The composition of partial functions \( f, g \in \mathbb{N} \rightarrow \mathbb{N} \) is the partial function

\[
g \circ f = \{(x, z) \mid (\exists y) (x, y) \in f \land (y, z) \in g \} \in \mathbb{N} \rightarrow \mathbb{N}
\]

Suppose \( F \) represents \( f \), \( G \) represents \( g \), and \( f \) and \( g \) are totally defined. Show that \( \lambda x. G(F x) \) represents \( g \circ f \). [2 marks]

(iii) Give an example to show that \( \lambda x. G(F x) \) need not represent \( g \circ f \) when \( f \) and \( g \) are not totally defined. [3 marks]