4 Computation Theory (AMP)

(*a*) (*i*) What does it mean for a λ-term to be a β-*normal form*? Defining the sets of canonical (C) and neutral (U) λ-terms by the grammar

$$C ::= \lambda x.\, C \mid U$$
$$U ::= x \mid U\, C$$

show that a λ-term is a β-normal form if and only if it is canonical. [5 marks]

(*ii*) Carefully stating any standard properties of β-reduction, explain why a λ-term reduces to at most one β-normal form (up to α-equivalence). [4 marks]

(*iii*) Give an example of a λ-term that does not reduce to any β-normal form. [2 marks]

(*b*) (*i*) Define what it means for a closed λ-term F to *represent* a partial function $f \in \mathbb{N} \rightharpoonup \mathbb{N}$. [4 marks]

(*ii*) The composition of partial functions $f, g \in \mathbb{N} \rightharpoonup \mathbb{N}$ is the partial function $g \circ f = \{(x, z) \mid (\exists y)\ (x, y) \in f \wedge (y, z) \in g\} \in \mathbb{N} \rightharpoonup \mathbb{N}$. Suppose F represents f, G represents g, and f and g are totally defined. Show that $\lambda x.\, G\,(F\, x)$ represents $g \circ f$. [2 marks]

(*iii*) Give an example to show that $\lambda x.\, G\,(F\, x)$ need not represent $g \circ f$ when f and g are not totally defined. [3 marks]