# COMPUTER SCIENCE TRIPOS Part IA

## NATURAL SCIENCES TRIPOS Part IA (Paper CS/1)

## POLITICS, PSYCHOLOGY, AND SOCIOLOGY TRIPOS Part I (Paper 9)

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Monday 3 June 2013 1.30 to 4.30

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COMPUTER SCIENCE Paper 1

*Answer **five** questions.*

*At least **one** question from each section is to be answered.*

*Submit the answers in five **separate** bundles, each with its own cover sheet. On each cover sheet, write the numbers of **all** attempted questions, and circle the number of the question attached.*

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

STATIONERY REQUIREMENTS
*Script paper*
*Blue cover sheets*
*Tags*

SPECIAL REQUIREMENTS
*Approved calculator permitted*

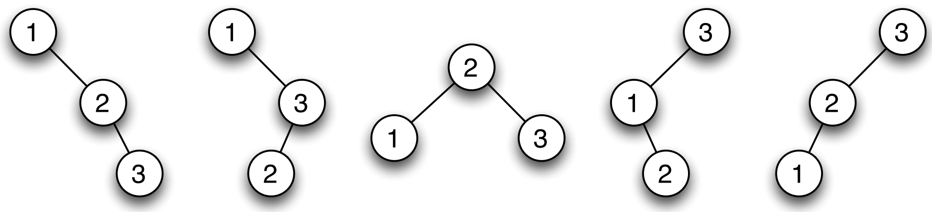
## SECTION A

### 1 Foundations of Computer Science

(*a*) Write brief notes on ML datatypes and pattern-matching in function declarations. [6 marks]

(*b*) A binary tree is either a *leaf* (containing no information) or is a *branch* containing a label and two subtrees (called the *left* and *right* subtrees). Write ML code for a function that takes a label and two lists of trees, returning all trees that consist of a branch with the given label, with the left subtree taken from the first list of trees and the right subtree taken from the second list of trees. [6 marks]

(*c*) Write ML code for a function that, given a list of distinct values, returns a list of all possible binary trees whose labels, enumerated in inorder, match that list. For example, given the list `[1,2,3]` your function should return (in any order) the following list of trees:




[8 marks]

All ML code must be explained clearly and should be free of needless complexity.

## 2 Foundations of Computer Science

The function perms returns all $n!$ permutations of a given $n$-element list.

```
fun cons x y = x::y;

fun perms [] = [[]]
  | perms xs =
      let fun perms1 ([],ys) = []
              | perms1 (x::xs,ys) =
                  map (cons x) (perms (rev ys @ xs)) @
                  perms1 (xs,x::ys)
      in  perms1 (xs,[])  end;
```

(*a*) Explain the ideas behind this code, including the function `perms1` and the expression `map (cons x)`. What value is returned by `perms [1,2,3]`? [7 marks]

(*b*) A student modifies `perms` to use an ML type of lazy lists, where `appendq` and `mapq` are lazy list analogues of `@` and `map`.

```
fun lperms [] = Cons ([], fn() => Nil)
  | lperms xs =
      let fun perms1 ([],ys) = Nil
              | perms1 (x::xs,ys) =
            appendq (mapq (cons x) (lperms (rev ys @ xs)),
                     perms1 (xs,x::ys))
      in  perms1 (xs,[])  end;
```

Unfortunately, `lperms` computes all $n!$ permutations as soon as it is called. Describe how lazy lists are implemented in ML and explain why laziness is not achieved here. [5 marks]

(*c*) Modify the function `lperms`, without changing its type, so that it computes permutations upon demand rather than all at once. [8 marks]

All ML code must be explained clearly and should be free of needless complexity.

## SECTION B

### 3 Discrete Mathematics I

(*a*) Consider the following assertions about the sets $A$, $B$ and $C$. Write them down in the language of predicate logic. Use only the constructions of predicate logic ($\forall$, $\exists$, $\neg$, $\Rightarrow$, $\wedge$, $\vee$) and the element-of symbol ($\in$). Do *not* use derived notions ($\cap$, $\cup$, $=$, etc.).

*Example:* "$A$ is a subset of $B$" can be formalized as $\forall x.\, x \in A \implies x \in B$.

(*i*) The sets $A$ and $B$ are equal.

(*ii*) Every element of $A$ is in the set $B$ or the set $C$.

(*iii*) If $A$ is disjoint from $B$ then $B$ and $C$ overlap.

[6 marks]

(*b*) State the principle of induction over lists. Use the language of predicate logic.
[2 marks]

(*c*) Consider the following functions over lists of integers, written in ML syntax.

```
fun app([],ys) = ys
  | app(x::xs,ys) = x::app(xs,ys);

fun rev([]) = []
  | rev(x::xs) = app(rev(xs),x::[]);

fun revapp([],ys) = ys
  | revapp(x::xs,ys) = revapp(xs,x::ys);
```

Prove that

$$\forall \texttt{xs}.\ \texttt{revapp(xs,[])} = \texttt{rev(xs)}$$

Your proof should be clear but it does not need to be a structured proof. You may use the abbreviation `xs @ ys` for `app(xs,ys)`. You may assume the following facts.

$$\forall \texttt{xs}.\ \texttt{xs @ []} = \texttt{xs} \qquad \forall \texttt{xs},\texttt{ys},\texttt{zs}.\ \texttt{xs @ (ys @ zs)} = \texttt{(xs @ ys) @ zs}$$

Hint: first use induction to show that

$$\forall \texttt{xs}.\ \forall \texttt{ys}.\ \texttt{revapp(xs,ys)} = \texttt{app(rev(xs),ys)}.$$

[12 marks]

## 4 Discrete Mathematics I

(*a*) Write down the introduction and elimination rules for the universal quantifier ($\forall$), the existential quantifier ($\exists$) and negation ($\neg$) in structured proof.

[6 marks]

(*b*) Write down the introduction rule for implication ($\Longrightarrow$) in structured proof.

[1 mark]

(*c*) Write down a structured proof of the following sentence.

$$(\forall x.\, \neg P(x)) \implies \neg \exists x.\, P(x)$$

[5 marks]

(*d*) Write down a structured proof of the following sentence. Clearly state any proof rules that you use in addition to those included in part (*a*) and part (*b*).

$$(\neg \forall x.\, \neg P(x)) \implies \exists x.\, P(x)$$

[8 marks]

## SECTION C

### 5 Algorithms I

One of several ways to perform string matching efficiently is with a finite state automaton (FSA).

(*a*) Give a brief but clear explanation of the FSA string matching algorithm, its complexity and any associated data structures. [*Note:* pseudocode of up to 10 lines is allowed, but not required.] [4 marks]

(*b*) Build the FSA that will find matches of the pattern $P = \texttt{pepep}$ in an arbitrary string $T$ over the alphabet $\{\texttt{e}, \texttt{o}, \texttt{p}\}$, explaining what you do and why. [6 marks]

(*c*) The correctness proof of the FSA string matching algorithm involves the function $\sigma_P(x)$, which is parametric in the pattern $P$ and takes as input a string $x$. Define $\sigma_P(x)$, explaining what it returns. [1 mark]

(*d*) Let $A, B, C, D$ be character strings; let $|A|$ be the length of string $A$; let $+$ denote integer addition or string concatenation depending on its operands. Let $D$ be the longest suffix of $A$ that is a prefix of $B$.

For each of the following claims: either prove the claim correct, or give a counterexample that proves it is incorrect. You may draw an explanatory picture if it helps clarity.

(*i*) $\sigma_B(A) = D$ [3 marks]

(*ii*) $\sigma_B(A + C) = |D| + |C|$ [3 marks]

(*iii*) $|C| = 1 \quad \Rightarrow \quad \sigma_B(A + C) = \sigma_B(A) + 1$ [3 marks]

## 6 Algorithms I

A *palindrome* is a string that, if reversed, remains the same, for example "madamimadam". A *subsequence* of a string $x$ is one obtained by dropping zero or more characters from $x$ and taking the remaining ones in order: for example "tan" is a subsequence of "pen**ta**go**n**". In this question you must find the longest palindrome subsequence (LPS) of a given string. [Note that the LPS may not be unique.]

(*a*) Explain why it is possible to apply dynamic programming to the LPS problem. Develop and explain a recursive equation for the length of the LPS. [6 marks]

(*b*) Develop and describe in detail, with pictures where appropriate, a bottom-up dynamic programming algorithm to solve the LPS problem. Include an explanation of how to recover the LPS from the bottom-up table you build. If you use pseudocode (not required), keep each pseudocode chunk under 10 lines and comment it clearly. Incomprehensible code will be scored as wrong. [9 marks]

(*c*) Derive the asymptotic worst-case running time of your algorithm. [2 marks]

(*d*) What else would you have to do to recover *all* the LPSs of a given string? [3 marks]

## SECTION D

### 7 Floating-Point Computation

The following two functions are algorithms for exponentiation where $x$ is a single-precision floating-point value and $n$ is an integer,

```
fun power1(x, n) = if n=0 then 1.0 else x * power1(x, n-1)

fun power2(x, n) = if n=0 then 1.0
                   else if even n then power2(x * x, n div 2)
                   else x * power2(x, n-1)
```

(*a*) What is, roughly, the largest value of $n$ that can be used without overflow when $x$ is 10.0? [1 mark]

(*b*) Suppose $x$ is close to 1.0.

(*i*) What is the worst possible relative error to expect in the answer from `power1` when $n = 100$? [3 marks]

(*ii*) Can we say anything useful about the absolute error in part (*b*)(*i*)? [1 mark]

(*iii*) What is the expected value of the relative error in results from `power1`? [1 mark]

(*c*) Sometimes the expected magnitude of error can be estimated as the result of a random walk.

(*i*) Under what conditions is this appropriate? [2 marks]

(*ii*) What is the random walk estimate for the relative error in part (*b*)(*i*)? [3 marks]

(*d*) If $x$ is again close to 1.0, what is the worst possible relative error to expect from `power2` when $n = 100$? [6 marks]

(*e*) For what range or class of $x$ values will `power2` with $n = 100$ give a result with no error? [3 marks]

## 8 Object-Oriented Programming with Java

*Sparse* matrices are matrices whose elements are predominantly zero. This question develops a Java representation for them called `SparseMatrix`.

The code below seeks to use an `ArrayList` of `LinkedList`s to implement the concept efficiently. It defines a class `Element` to store the column number and value for an element. Each row is represented by a `LinkedList` of `Element`s with non-zero values only. Few, if any, rows are all zeros and so the `ArrayList` is used to store a `LinkedList` for every row in ascending row order.

```
public class Element {
    public int column;
    public int value;
}

public class SparseMatrix {
  private int mRows;    // Number of rows
  private int mCols;    // Number of columns
  private ArrayList<LinkedList<Element> > mMatrix;    // Data

}
```

(*a*) Give two reasons why `Element` should not have `public` state and provide a better mutable `Element` definition. [4 marks]

(*b*) Explain why `ArrayList` and `LinkedList` are appropriate choices in this context. [2 marks]

(*c*) Write a constructor for `SparseMatrix` that takes arguments specifying the number of rows and columns and initialises state appropriately. [2 marks]

(*d*) Provide the member method `get(int r, int c)`, which retrieves the value at row `r` and column `c` of the matrix, and the method `set(int r, int c, int v)`, which sets the value of it to `v`. Your methods should throw an exception if invalid arguments are supplied. [6 marks]

(*e*) By making `Element` objects `Comparable` show how to keep the linked lists in ascending column order and hence how to make `get()` and `set()` more efficient. If `get()` operations are more common than `set()` operations, suggest a better choice than `LinkedList` for the type of the inner list. [6 marks]

**END OF PAPER**