COMPUTER SCIENCE TRIPOS Part IB

Thursday 7 June 2012 1.30 to 4.30

COMPUTER SCIENCE Paper 6

*Answer **five** questions.*

*Submit the answers in five **separate** bundles, each with its own cover sheet. On each cover sheet, write the numbers of **all** attempted questions, and circle the number of the question attached.*

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

STATIONERY REQUIREMENTS
Script paper
Blue cover sheets
Tags

SPECIAL REQUIREMENTS
Approved calculator permitted

1 Complexity Theory

(*a*) Suppose L_1 and L_2 are languages in **P**. What can you say about the complexity of each of the following? Justify your answer in each case.

(*i*) $L_1 \cup L_2$. [3 marks]

(*ii*) $L_1 \cap L_2$. [3 marks]

(*iii*) The complement of L_1. [2 marks]

(*b*) Suppose L_1 and L_2 are languages in **NP**. What can you say about the complexity of each of the following? Justify your answer in each case.

(*i*) $L_1 \cup L_2$. [3 marks]

(*ii*) $L_1 \cap L_2$. [3 marks]

(*iii*) The complement of L_1. [2 marks]

(*c*) Give an example of a language in **NP** that is *not* **NP**-complete and prove that it is not. [4 marks]

2 Complexity Theory

(*a*) Consider the following decision problem.

> Given positive integers $x_1, \ldots, x_n, y$, determine whether there is a set $I \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in I} x_i = y$.

(*i*) Prove that, if the integers $x_1, \ldots, x_n, y$ are written in *unary*, then the problem is in **P**. [*Hint:* consider a recursive algorithm which checks whether either of y or $y - x_1$ can be expressed as the sum of a subset of $x_2, \ldots, x_n$.] [6 marks]

(*ii*) What can you say about the complexity of the decision problem when the integers $x_1, \ldots, x_n, y$ are written in *binary*? You do not need to prove your answer, but state clearly any standard results you use. [2 marks]

(*b*) What does it mean for a language L to be **NP**-hard? What does it mean for L to be **NP**-complete? [2 marks]

(*c*) We write $[M]$ to be the string encoding a Turing machine M using some standard coding scheme. Consider the language A defined by:

$$A = \{[M], x \mid M \text{ accepts } x\}$$

where "$[M], x$" denotes the string $[M]$ followed by a comma and then x.

(*i*) Prove that A is **NP**-hard. [8 marks]

(*ii*) Is A **NP**-complete? Justify your answer. [2 marks]

3 Computation Theory

(*a*) Define what is a *Turing machine* and a *Turing machine computation*. [7 marks]

(*b*) What is meant by a *partial function* from $\mathbb{N}^n$ to $\mathbb{N}$? Define what it means for such a partial function to be *Turing computable*. [4 marks]

(*c*) Describe the *Church-Turing Thesis* and some evidence for its truth. [4 marks]

(*d*) Assuming the existence of a universal *register* machine, give an example, with justification, of a partial function that is not Turing computable. [5 marks]

4 Computation Theory

(*a*) Define what it means for a set of numbers $S \subseteq \mathbb{N}$ to be register machine *decidable*. Why are there only countably many such sets? Deduce the existence of a set of numbers that is not register machine decidable. (Any standard results that you use should be clearly stated.) [4 marks]

(*b*) A set of numbers $S \subseteq \mathbb{N}$ is said to be *computably enumerable* if either it is empty or equal to $\{f(x) \mid x \in \mathbb{N}\}$ for some total function $f : \mathbb{N} \to \mathbb{N}$ that is register machine computable.

(*i*) Show that if S is register machine decidable, then it is computably enumerable. [*Hint:* consider separately the cases when S is, or is not empty.] [4 marks]

(*ii*) Show that if both S and its complement $\{x \in \mathbb{N} \mid x \notin S\}$ are computably enumerable, then S is register machine decidable. [*Hint:* consider a register machine that interleaves the enumeration of S and its complement.] [6 marks]

(*c*) Let $\varphi_e : \mathbb{N} \rightharpoonup \mathbb{N}$ denote the partial function computed by the register machine with code $e \in \mathbb{N}$ and consider the set $T = \{e \in \mathbb{N} \mid \varphi_e \text{ is a total function}\}$.

(*i*) Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a register machine computable total function such that $f(x) \in T$ for all $x \in \mathbb{N}$. Define $\hat{f}(x)$ to be $\varphi_{f(x)}(x) + 1$. Show that $\hat{f} = \varphi_e$ for some $e \in T$. [3 marks]

(*ii*) Deduce that T is not computably enumerable. [3 marks]

5 Logic and Proof

(*a*) Exhibit a model for the following set of formulas, or prove that none exists.

$$P \to Q \wedge R \qquad P \wedge Q \to S \qquad \neg R \leftrightarrow S \qquad P \vee Q$$

[8 marks]

(*b*) Consider the following set of clauses:

$$\{\neg(x < y),\ -y < -x\} \qquad \{\neg(x < y),\ x + z < y + z\} \qquad \{0 < 1\}$$

(*i*) What is the Herbrand universe of these clauses? [3 marks]

(*ii*) What semantics must any Herbrand interpretation of the clauses attach to the function symbols? [3 marks]

(*iii*) Specify an Herbrand model by giving a semantics of the relation $<$, justifying your choice with reference to a natural model of the set of clauses. [6 marks]

6 Logic and Proof

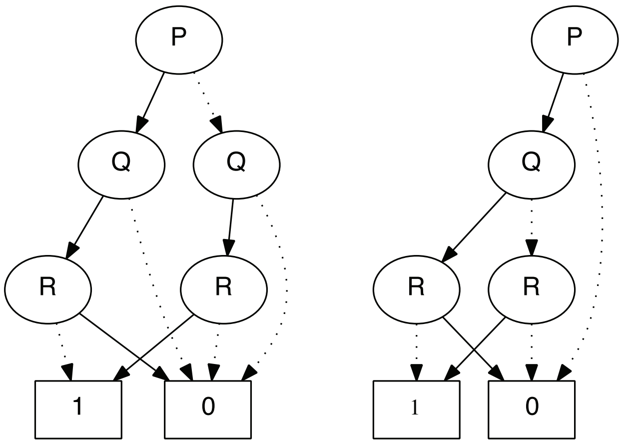
(*a*) Demonstrate the sequent calculus, the free-variable tableau calculus and resolution by using each of them to prove the following formula:

$$(P(a,b) \lor \exists z\, P(z,z)) \to \exists x \exists y\, P(x,y)$$

Comment briefly on the similarities and differences among these three methods. [12 marks]

(*b*) Prove $\Box\Diamond P \to \Diamond\Box P$ using the sequent calculus for S4 modal logic, or exhibit a falsifying interpretation. [4 marks]

(*c*) Briefly outline the procedure for converting a formula to a BDD, illustrating your answer by constructing the BDD that represents the conjunction of those below. [4 marks]



7 Mathematical Methods for Computer Science

(*a*) Define linear independence and linear dependence for the set of vectors $\{v_1, v_2, \ldots, v_n\}$ of a vector space V over a field $\mathbb{F}$ of scalars $a_1, a_2, \ldots, a_n \in \mathbb{F}$.
[4 marks]

(*b*) Using the Euclidean norm on an inner product space $V = \mathbb{R}^3$, for the following vectors $u, v \in V$ whose span is a linear subspace of V,

$$u = \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}} \right)$$

$$v = \left(\sqrt{3},\ -\frac{\sqrt{3}}{2},\ -\frac{\sqrt{3}}{2} \right)$$

demonstrate whether u, v form an *orthogonal system*, and also whether they form an *orthonormal system*. [4 marks]

(*c*) Using a diagram in the complex plane showing the N^{th} roots of unity, explain why all the values of complex exponentials that are needed for computing the Discrete Fourier transform of N data points are powers of a primitive N^{th} root of unity (circled here for $N = 16$), and explain why such factorisation greatly reduces the number of multiplications required in a Fast Fourier transform.
[4 marks]

(*d*) For the function $f(x) = e^{-a|x|}$ with $a > 0$, derive its Fourier transform $F(\omega)$.
[4 marks]

(*e*) For a function $f(x)$ whose Fourier transform is $F(\omega)$, what is the Fourier transform of $f^{(n)}(x)$, the n^{th} derivative of $f(x)$ with respect to x? Explain how Fourier methods make it possible to define non-integer orders of derivatives, and name one scientific field in which it is useful to take half-order derivatives.
[4 marks]

8 Mathematical Methods for Computer Science

(*a*) Consider the Markov Chain, X_n, on the states $i = 0, 1, 2, \ldots$ with transition matrix given by

$$p_{i,i-1} = p \qquad i = 1, 2, \ldots$$

$$p_{i,i+1} = 1 - p \qquad i = 0, 1, \ldots$$

$$p_{0,0} = p$$

where $0 < p < 1$.

(*i*) Show that the Markov chain is irreducible. [2 marks]

(*ii*) Show that the Markov chain is aperiodic. [2 marks]

(*iii*) Find a condition on p to make the Markov chain positive recurrent and find the stationary distribution in this case. [6 marks]

(*b*) Consider the PageRank algorithm.

(*i*) Describe PageRank as a Markov chain model for the motion between nodes in a graph where the nodes correspond with web pages. [5 marks]

(*ii*) Explain the main mathematical results that underpin the relevance of PageRank to a notion of web page *importance*. [5 marks]

9 Semantics of Programming Languages

This question is about a very simple programming language with support for exceptions. The syntax and semantics is given to you.

The language has the following syntax.

Exceptions: $\gamma, \gamma_1, \gamma_2, \ldots$

Expressions: $e ::= \mathsf{true} \mid \mathsf{false} \mid \mathsf{if}\ e_1\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3$
$\quad \mid e_1\ \mathsf{handle}\ \gamma \Rightarrow e_2 \mid \mathsf{raise}\,\gamma$

Types: There is only one type: bool

A *typing context* is a set Γ of exceptions. The typing rules are as follows:

$$\frac{}{\Gamma \vdash \mathsf{true} : \mathsf{bool}} \qquad \frac{}{\Gamma \vdash \mathsf{false} : \mathsf{bool}} \qquad \frac{\Gamma \vdash e_1 : \mathsf{bool} \quad \Gamma \vdash e_2 : \mathsf{bool} \quad \Gamma \vdash e_3 : \mathsf{bool}}{\Gamma \vdash \mathsf{if}\ e_1\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3 : \mathsf{bool}}$$

$$\frac{\Gamma \cup \{\gamma\} \vdash e_1 : \mathsf{bool} \quad \Gamma \vdash e_2 : \mathsf{bool}}{\Gamma \vdash e_1\ \mathsf{handle}\ \gamma \Rightarrow e_2 : \mathsf{bool}}\ (\gamma \notin \Gamma) \qquad \frac{}{\Gamma \vdash \mathsf{raise}\,\gamma : \mathsf{bool}}\ (\gamma \in \Gamma)$$

The reduction relation $\longrightarrow$ is defined as follows:

$$\frac{}{\mathsf{if\ true\ then}\ e_2\ \mathsf{else}\ e_3 \longrightarrow e_2} \qquad \frac{}{\mathsf{if\ false\ then}\ e_2\ \mathsf{else}\ e_3 \longrightarrow e_3}$$

$$\frac{e_1 \longrightarrow e_1'}{\mathsf{if}\ e_1\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3 \longrightarrow \mathsf{if}\ e_1'\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3}$$

$$\frac{}{(\mathsf{true\ handle}\ \gamma \Rightarrow e_2) \longrightarrow \mathsf{true}} \qquad \frac{}{(\mathsf{false\ handle}\ \gamma \Rightarrow e_2) \longrightarrow \mathsf{false}}$$

$$\frac{}{((\mathsf{raise}\,\gamma)\ \mathsf{handle}\ \gamma \Rightarrow e_2) \longrightarrow e_2} \qquad \frac{e_1 \longrightarrow e_1'}{(e_1\ \mathsf{handle}\ \gamma \Rightarrow e_2) \longrightarrow (e_1'\ \mathsf{handle}\ \gamma \Rightarrow e_2)}$$

$$\frac{}{\mathsf{if}\ (\mathsf{raise}\,\gamma)\ \mathsf{then}\ e_2\ \mathsf{else}\ e_3 \longrightarrow \mathsf{raise}\,\gamma} \qquad \frac{}{((\mathsf{raise}\,\gamma)\ \mathsf{handle}\ \gamma' \Rightarrow e_2) \longrightarrow \mathsf{raise}\,\gamma}\ (\gamma \neq \gamma')$$

(*a*) Consider the program $e_0 \stackrel{\text{def}}{=} \mathsf{if}\ ((\mathsf{raise}\,\gamma)\ \mathsf{handle}\ \gamma \Rightarrow \mathsf{true})\ \mathsf{then\ false\ else\ true}$.

(*i*) Give a derivation for $\vdash e_0 : \mathsf{bool}$. [4 marks]

(*ii*) Give a derivation for all of the transition steps for e_0. [3 marks]

(*b*) Prove the following theorem about this language:

If $\Gamma \vdash e : \mathsf{bool}$ then one of the following four conditions holds: (1) $e = \mathsf{true}$; (2) $e = \mathsf{false}$; (3) there is $\gamma \in \Gamma$ such that $e = (\mathsf{raise}\,\gamma)$; (4) there is e' such that $e \longrightarrow e'$. [13 marks]

10 Semantics of Programming Languages

This question is about a simple functional programming language with the following syntax.

Expressions: $e ::= x \mid \mathsf{skip} \mid \mathsf{fn}\, x : T \Rightarrow e \mid e\, e'$

Types: $T ::= \mathsf{unit} \mid T \to T'$

(*a*) Give rules defining a typing relation ($\vdash$) for this language. [5 marks]

(*b*) Give a brief illustration of the following concepts: *free variables* and *closed expression*. [2 marks]

(*c*) Give rules defining a transition relation ($\longrightarrow$) for this language. Use the call-by-value evaluation order, and take care to say what the values are. [5 marks]

(*d*) State and prove a Type Progress theorem for this language. [8 marks]

END OF PAPER