2011 Paper 9 Question 3

Denotational Semantics

(*a*) A PCF type τ is said to be *finite* (resp. *infinite)* if the domain $[\![\tau]\!]$ is finite (resp. infinite). An element d of the domain $[\![\tau]\!]$ is said to be *definable* whenever there exists a closed PCF term $M : \tau$ such that $[\![M]\!] = d$.

Indicate whether the following statements are true or false. Provide an argument for each answer. You may use standard results provided that you state them clearly.

(*i*) For all finite PCF types τ, every element of the domain $[\![\tau]\!]$ is definable. [5 marks]

(*ii*) For all infinite PCF types τ, every element of the domain $[\![\tau]\!]$ is definable. [5 marks]

(*b*) Consider the following two statements for PCF terms M_1 and M_2 for which the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold for some type environment Γ and type τ.

(1) For all PCF contexts $\mathcal{C}[-]$ for which $\mathcal{C}[M_1] : \mathit{bool}$ and $\mathcal{C}[M_2] : \mathit{bool}$,

$$\mathcal{C}[M_1] \Downarrow_{bool} \iff \mathcal{C}[M_2] \Downarrow_{bool}$$

where, for $M : \tau$, the notation $M \Downarrow_\tau$ stands for the existence of a value $V : \tau$ for which $M \Downarrow_\tau V$.

(2) For all PCF contexts $\mathcal{C}[-]$ for which $\mathcal{C}[M_1] : \mathit{bool}$ and $\mathcal{C}[M_2] : \mathit{bool}$,

$$\mathcal{C}[M_1] \Downarrow_{bool} \mathbf{true} \iff \mathcal{C}[M_2] \Downarrow_{bool} \mathbf{true}$$

(*i*) Show that (1) implies (2). [5 marks]

(*ii*) Define the notion of contextual equivalence in PCF and show that (2) implies that M_1 and M_2 are contextually equivalent. [5 marks]