2011 Paper 2 Question 6

Discrete Mathematics II

Let E be a set. Assume $\mathcal{F} \subseteq \mathcal{P}(E)$ satisfies the two conditions

1. $\forall X \subseteq \mathcal{F}.\ \bigcup X \in \mathcal{F}$
2. $\forall X \subseteq \mathcal{F}.\ \bigcap X \in \mathcal{F}$

Recall

$$\bigcup X =_{\text{def}} \{e \in E \mid \exists x \in X.\ e \in x\} \quad \text{and} \quad \bigcap X =_{\text{def}} \{e \in E \mid \forall x \in X.\ e \in x\}$$

(a) Explain why $\emptyset \in \mathcal{F}$ and $E \in \mathcal{F}$. [2 marks]

(b) Define the binary relation $\lesssim$ on E by

$$e' \lesssim e \ \text{ iff } \ \forall x \in \mathcal{F}.\ e \in x \ \Rightarrow e' \in x$$

for $e, e' \in E$. State clearly what it would mean for $\lesssim$ to be reflexive and transitive. Show $\lesssim$ is reflexive and transitive. [5 marks]

(c) For $e \in E$, define

$$[e] = \bigcap \{x \in \mathcal{F} \mid e \in x\}$$

Explain why $[e] \in \mathcal{F}$. Show

$$[e] = \{e' \mid e' \lesssim e\}$$

[6 marks]

(d) Say a subset z of E is *down-closed* iff

$$e' \lesssim e \ \&\ e \in z \quad \Rightarrow \quad e' \in z$$

for all $e, e' \in E$. Show $\mathcal{F}$ consists of precisely the down-closed subsets of E by showing:

(i) any $x \in \mathcal{F}$ is down-closed; [3 marks]

(ii) for any down-closed subset z of E,

$$z = \bigcup \{[e] \mid e \in z\}$$

and hence $z \in \mathcal{F}$ (why?). [4 marks]