2011 Paper 2 Question 5

Discrete Mathematics II

Let A and B be sets. Let $F \subseteq \mathcal{P}(A) \times B$. So a typical element of F is a pair (X, b) where $X \subseteq A$ and $b \in B$. Define the function

$$f : \mathcal{P}(A) \to \mathcal{P}(B)$$

to be such that

$$f(x) = \{b \mid \exists X \subseteq x.\ (X, b) \in F\}$$

for $x \in \mathcal{P}(A)$.

(*a*) Show

$$\text{if } x \subseteq y \text{ then } f(x) \subseteq f(y)$$

for all $x, y \in \mathcal{P}(A)$. [3 marks]

(*b*) Suppose

$$x_0 \subseteq x_1 \subseteq \cdots \subseteq x_n \subseteq \cdots$$

is a chain of subsets in $\mathcal{P}(A)$. Recall $\bigcup_{n \in \mathbb{N}_0} x_n = \{a \mid \exists n \in \mathbb{N}_0.\ a \in x_n\}$. Show that

$$\bigcup_{n \in \mathbb{N}_0} f(x_n) \subseteq f(\bigcup_{n \in \mathbb{N}_0} x_n)$$

[Hint: Use part (*a*).] [4 marks]

(*c*) Assume now that $F \subseteq \mathcal{P}_{\text{fin}}(A) \times B$ where $\mathcal{P}_{\text{fin}}(A)$ consists of the finite subsets of A. So now a typical element of F is a pair (X, b) where X is a *finite* subset of A and $b \in B$. Suppose $x_0 \subseteq x_1 \subseteq \cdots \subseteq x_n \subseteq \cdots$ is a chain of subsets in $\mathcal{P}(A)$. Show that

$$f(\bigcup_{n \in \mathbb{N}_0} x_n) \subseteq \bigcup_{n \in \mathbb{N}_0} f(x_n)$$

Deduce

$$f(\bigcup_{n \in \mathbb{N}_0} x_n) = \bigcup_{n \in \mathbb{N}_0} f(x_n) \qquad (\dagger)$$

[6 marks]

(*d*) Show that (†) need not hold if the set X in elements (X, b) of F is infinite. [7 marks]