2010 Paper 6 Question 8

Mathematical Methods for Computer Science

(*a*) Let $\{X_n : n = 0, 1, \cdots\}$ be a two-state Markov chain with transition probabilities given by the matrix

$$P = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}$$

Let $N_{i,j} = \mathbb{E}$ (number of visits to state j before first return to state $i | X_0 = i$) for $i \neq j$. Prove that

$$N_{2,1} = \frac{1-q}{1-p}$$

giving careful attention to any special cases.

[Hint: Consider $\phi_{i,j}^{(n)} = \mathbb{P}(X_1 = j, X_2 = j, \ldots, X_n = j, X_{n+1} = i | X_0 = i)$.]
[8 marks]

(*b*) Two marksmen, Alice and Bob, take turns shooting at a target. They agree that Alice will shoot after each hit, while Bob will shoot after each miss. Suppose Alice hits the target with probability α, while Bob hits the target with probability β. Over a long period of time, what proportion of shots hit the target? State carefully any theorems that you use in arriving at your answer. Again, check any special cases. [12 marks]