2010 Paper 6 Question 7

Mathematical Methods for Computer Science

(*a*) What is an orthonormal basis? Why is it important that a basis be orthonormal? [4 marks]

(*b*) A real, periodic function, $f(x)$, can be expressed as a Fourier series. This can be shown in several ways. One is as a sum of weighted, offset, cosine functions:

$$f(x) = \sum_{k=0}^{\infty} A_k \cos\left(xk\frac{2\pi}{T} - \theta_k\right)$$

A second way is as a sum of complex exponentials with complex coefficients:

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \exp\left(ixk\frac{2\pi}{T}\right)$$

where the complex coefficients, c_k, have the constraint $c_k = c^*_{-k}$ for $f(x)$ real.

(*i*) Prove that these two alternative expressions of the Fourier series are equivalent. [8 marks]

(*ii*) Express the complex coefficient c_k in terms of the real parameters A_k and θ_k. [2 marks]

(*c*) Consider the box function:

$$b(x) = \begin{cases} 1, & |x| \le \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

and the tent function:

$$t(x) = b(x) * b(x) = \begin{cases} x+1, & -1 \le x < 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(*i*) Find the Fourier transform of $b(x)$. [4 marks]

(*ii*) Find the Fourier transform of $t(x)$. [2 marks]

The following formulæ may be useful.

$$\begin{aligned} e^{i\phi} &= \cos\phi + i\sin\phi \\ \sin(a+b) &= \sin a\cos b + \cos a\sin b \\ \sin(a-b) &= \sin a\cos b - \cos a\sin b \\ \cos(a+b) &= \cos a\cos b - \sin a\sin b \\ \cos(a-b) &= \cos a\cos b + \sin a\sin b \\ \cos^2\phi + \sin^2\phi &= 1 \end{aligned}$$