Semantics of Programming Languages

Below is the syntax and operational semantics for a pure functional language.

Types: \[ T ::= \text{bool} \mid T \rightarrow T \]

Variables: \( \{x, y, z, \ldots \} \)

Expressions: \[ e ::= \text{true} \mid \text{false} \mid \text{if } e \text{ then } e_1 \text{ else } e_2 \mid \text{fn}(x : T) \Rightarrow e \mid e e'. \]

In the expression fn\((x : T) \Rightarrow e\), the variable \(x\) is binding in \(e\).

\[
\begin{align*}
\text{(if1)} & \quad \text{(if true then } e_1 \text{ else } e_2) \rightarrow e_1 \\
\text{(if2)} & \quad \text{(if false then } e_1 \text{ else } e_2) \rightarrow e_2 \\
\text{(if3)} & \quad e \rightarrow e' \\
\text{(app)} & \quad \frac{e_1 \rightarrow e_1'}{e_1 e_2 \rightarrow e_1' e_2} \\
\text{(fn)} & \quad \frac{e \rightarrow e'}{(\text{fn}(x : T) \Rightarrow e) e \rightarrow \{e'/x\} e}
\end{align*}
\]

(There is no need for a store because there are no store access operations.)

(a) Is this a call-by-value or a call-by-name language? Revise the operational semantics to demonstrate the other calling convention. [4 marks]

(b) A type environment is a finite partial function \(\Gamma\) from variables to types. Define a typing relation \(\Gamma \vdash e : T\) by giving a set of rules. [6 marks]

(c) Are the following expressions typable?

\[
e_1 = \text{fn}(f : (\text{bool} \rightarrow \text{bool}) \rightarrow \text{bool}) \Rightarrow (\text{fn}(f : \text{bool} \rightarrow \text{bool}) \Rightarrow f f)
\]
\[
e_2 = \text{fn}(f : (\text{bool} \rightarrow \text{bool}) \rightarrow (\text{bool} \rightarrow \text{bool})) \Rightarrow (\text{fn}(x : \text{bool}) \Rightarrow (f x) x)
\]

[2 marks]

(d) State formally the following two theorems of the one-step reduction semantics at the top of the page and the type system that you defined in part (b): Progress and Type Preservation. Take care to explain what a value is. (No proofs are required for this part.) [3 marks]

(e) State and prove the Type Safety theorem. You may use the results stated in part (d). [5 marks]