2010 Paper 2 Question 6

Discrete Mathematics II

A preorder $(E, \lesssim)$ consists of a set E on which there is a binary relation $\lesssim$ which is reflexive and transitive.

(*a*) A subset $x \subseteq E$ is *down-closed* iff

$$\forall e, e' \in E.\ e \in x \ \&\ e' \lesssim e \ \Rightarrow\ e' \in x$$

Write $\mathcal{D}(E)$ for the set of all down-closed subsets of E.

Show that

(*i*) $\emptyset \in \mathcal{D}(E)$, and [1 mark]

(*ii*) if $X \subseteq \mathcal{D}(E)$ then $\bigcup X \in \mathcal{D}(E)$. [3 marks]

[Recall $\bigcup X =_{\text{def}} \{e \in E \mid \exists x \in X.\, e \in x\}$.]

(*b*) An element $p \in \mathcal{D}(E)$ is said to be a *complete prime* iff

$$\forall X \subseteq \mathcal{D}(E).\ p \subseteq \bigcup X \ \Rightarrow\ \exists x \in X.\ p \subseteq x$$

For $e \in E$, define $[e]$ to be the down-closed set $\{e' \in E \mid e' \lesssim e\}$.

(*i*) Is $\emptyset$ a complete prime? Justify your answer. [3 marks]

(*ii*) Show for all $x \in \mathcal{D}(E)$ that

$$x = \bigcup \{[e] \mid e \in x\}$$

[3 marks]

(*c*) For $x \in \mathcal{D}(E)$, show that x is a complete prime iff $x = [e]$ for some $e \in E$. [Remember to show both directions of the "iff".] [6 marks]

(*d*) For $x \in \mathcal{D}(E)$, define

$$N(x) = \bigcup \{w \in \mathcal{D}(E) \mid w \cap x = \emptyset\}$$

Show, for $e \in E$, that

$$e \in N(x) \ \text{ iff } \ \forall e' \lesssim e.\ e' \notin x$$

[4 marks]