2009 Paper 4 Question 9

Mathematical Methods for Computer Science

For functions $f : \mathbb{R} \to \mathbb{C}$ define the *Fourier transform* of f, written $\mathcal{F}_{[f]}(\omega)$, as the function $\mathbb{R} \to \mathbb{C}$ given by

$$\mathcal{F}_{[f]}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

whenever the integral exists.

(*a*) Show that

$$\mathcal{F}_{[\alpha f + \beta g]}(\omega) = \alpha \mathcal{F}_{[f]}(\omega) + \beta \mathcal{F}_{[g]}(\omega)$$

for all complex constants $\alpha, \beta \in \mathbb{C}$ and functions f and g such that their Fourier transforms exist. [4 marks]

(*b*) For $a, b \in \mathbb{R}$ with $a \neq 0$, let $g(x) = f(ax + b)$. Show that

$$\mathcal{F}_{[g]}(\omega) = \frac{1}{|a|} e^{i\omega b/a} \mathcal{F}_{[f]}\left(\frac{\omega}{a}\right)$$

[6 marks]

(*c*) Derive $\mathcal{F}_{[f_c]}(\omega)$ where

$$f_c(x) = \begin{cases} 1 & -c < x \leq c \\ 0 & \text{otherwise} \end{cases}$$

and c is a positive constant. [4 marks]

(*d*) For the piecewise constant function

$$h(x) = \begin{cases} 0 & x > 4 \\ 2 & 3 < x \leq 4 \\ 1 & 2 < x \leq 3 \\ 3 & 1 < x \leq 2 \\ 0 & x \leq 1 \end{cases}$$

derive its Fourier transform $\mathcal{F}_{[h]}(\omega)$. [6 marks]