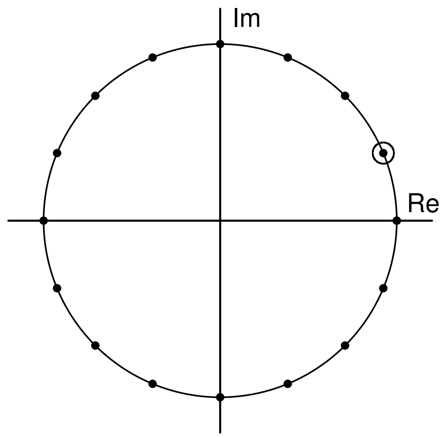
2008 Paper 9 Question 10

Information Theory and Coding



(*a*) Fast Fourier Transform algorithms use factorisation of discrete complex exponentials to avoid repeated multiplications by common factors. The diagram on the right shows a unit circle in the complex plane. The unit circle represents a continuous complex exponential (one orbit around it spans one cycle), and the 16 dots represent discrete samples of this Fourier component which need to be multiplied by 16 data points and summed to compute one discrete Fourier coefficient.

(*i*) The circled dot $\boxed{e^{2\pi i/n}}$ is a *primitive* n^{th}-root of unity, where for this diagram $n = 16$. Write down a similar expression for the full set of the n^{th}-roots of unity (all the dots), indexed by k where $1 \leq k \leq n$. [2 marks]

(*ii*) The 16 frequency components needed to compute the discrete Fourier transform of 16 data points are obtained by undersampling the dots; *e.g.* the 2^{nd} frequency uses every 2^{nd} dot and orbits twice. Explain the redundancy that occurs when multiplying these discrete complex exponentials by the data points. [5 marks]

(*iii*) For n data points, roughly how many multiplications are needed in a Fast Fourier Transform algorithm that avoids these redundancies? [2 marks]

(*b*) Explain the meaning of “self-Fourier,” and cite at least *two* examples of mathematical objects having this property. [3 marks]

(*c*) Consider an alphabet of 8 symbols whose probabilities are as follows:

A	B	C	D	E	F	G	H
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{128}$

(*i*) If someone has selected one of these symbols and you need to discover which symbol it is by asking “yes/no” questions that will be truthfully answered, what would be the most efficient sequence of such questions that you could ask in order to discover the selected symbol? [2 marks]

(*ii*) By what principle can you claim that each of your proposed questions is maximally informative? [2 marks]

(*iii*) On average, how many such questions will need to be asked before the selected symbol is discovered? [2 marks]

(*iv*) What is the entropy of the above symbol set? [2 marks]