2008 Paper 2 Question 4

Discrete Mathematics

Let I be a non-empty subset of the natural numbers $\mathbb{N} = \{1, 2, 3, \cdots\}$.

The set S is defined to be least subset of $\mathbb{N}$ such that

> $I \subseteq S$, and
> if $m, n \in S$ and $m < n$, then $(n - m) \in S$.

Define h to be the least member of S. This question guides you through to a proof that h coincides with the *highest common factor* of I, written $hcf(I)$, and defined to be the natural number with the properties that

> $hcf(I)$ divides n for every element $n \in I$, and
> if k is a natural number which divides n for every $n \in I$, then k divides $hcf(I)$.

[Throughout this question you may assume elementary facts about division.]

(*a*) The set S may also be described as the least subset of $\mathbb{N}$ closed under certain rules. Describe the rules. Write down a principle of rule induction appropriate for the set S. [4 marks]

(*b*) Show by rule induction that $hcf(I)$ divides n for every $n \in S$. [3 marks]

(*c*) Let $n \in S$. Establish that

$$\text{if } \; p.h < n \; \text{ then } \; (n - p.h) \in S$$

for all non-negative integers p. [5 marks]

(*d*) Show that h divides n for every $n \in S$. [Hint: suppose otherwise and derive a contradiction.] [5 marks]

(*e*) Explain very briefly why the results of parts (*b*) and (*d*) imply that $h = hcf(I)$. [3 marks]