

2007 Paper 9 Question 15

Denotational Semantics

Assume a denotational semantics is given for PCF: a mapping of PCF types τ to domains $\llbracket \tau \rrbracket$ together with a mapping of closed PCF terms of type τ , $M : \tau$, to elements of the corresponding domain $\llbracket M \rrbracket \in \llbracket \tau \rrbracket$. In the following you may refer to the notion of a PCF context without an explicit definition.

- (a) State clearly what properties the semantics must fulfil in order to be compositional. Give the fundamental property that a compositional semantics enjoys with respect to arbitrary contexts. [2 marks]
- (b) Assume the standard typed evaluation operational semantics for PCF. Without referring to explicit evaluation rules or to the syntax of PCF, state clearly what properties the denotational semantics must fulfil in order to be:
- (i) sound; [1 mark]
- (ii) adequate. [2 marks]
- (c) Define:
- (i) the contextual preorder \leq_{ctx} for closed PCF terms; [2 marks]
- (ii) contextual equivalence \cong_{ctx} for closed PCF terms. [1 mark]
- (d) Prove that if a denotational semantics is compositional, sound and adequate then denotational equality implies contextual equivalence. [4 marks]
- (e) State the extensionality properties of the contextual preorder for both base types (*bool* and *nat*) and function types. [4 marks]
- (f) Using extensionality, prove that the following terms are contextually equivalent:
- (i) $(\mathbf{fn} x : \mathit{bool} \rightarrow \mathit{bool}. \mathbf{0})(\mathbf{fn} x : \mathit{bool}. \mathbf{true})$ and $\mathbf{0}$; [2 marks]
- (ii) $\mathbf{fn} x : \mathit{nat}. \left((\mathbf{fn} y : \mathit{nat} \rightarrow \mathit{nat}. y(yx))(\mathbf{fn} z : \mathit{nat}. \mathbf{succ}(z)) \right)$
and
 $\mathbf{fn} x : \mathit{nat}. \mathbf{succ}(\mathbf{succ}(x))$. [2 marks]